Principles of Communications

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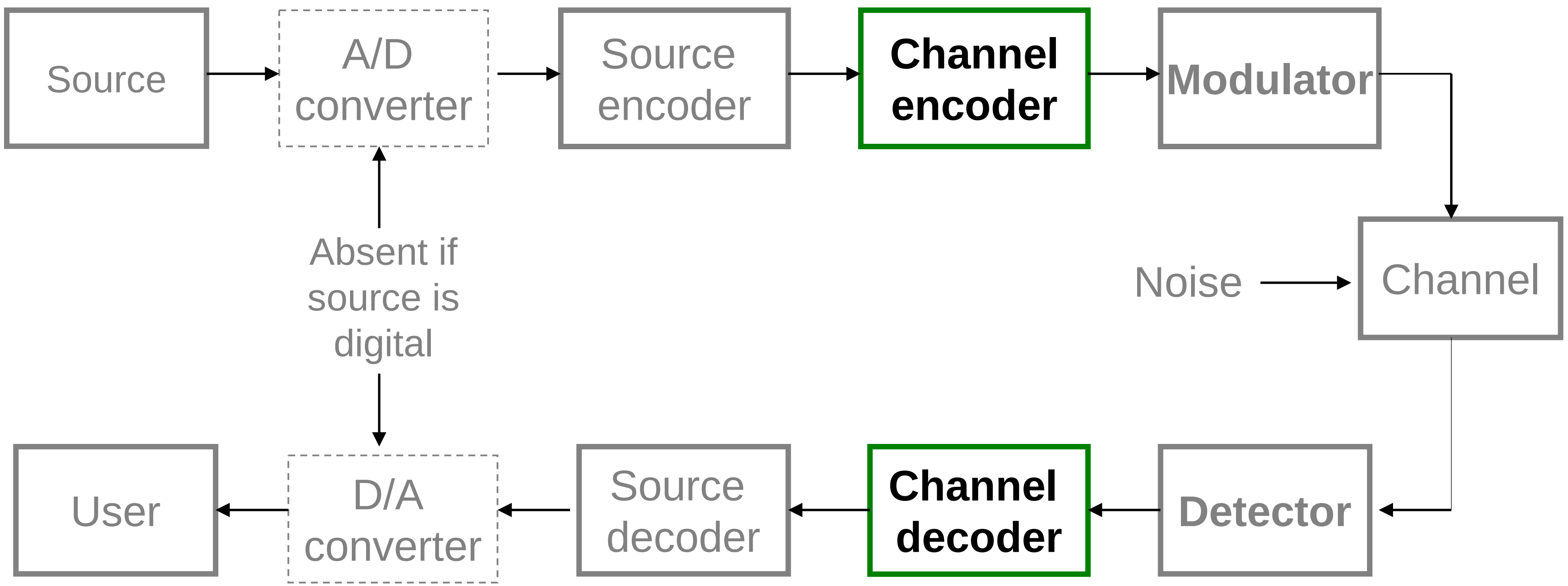
Chapter 10: Channel Coding

Selected from Chapter 13.1 – 13.3 of *Fundamentals of Communications Systems*, Pearson Prentice Hall 2005, by Proakis & Salehi

Topics to be Covered



- Linear block code (线性分组码)
- Convolutional code (卷积码)

Information Theory and Channel Coding

- Shannon's noisy channel coding theorem tells that adding controlled redundancy allows transmission at arbitrarily low bit error rate (BER) as long as R<=C
- Error control coding (ECC) uses this controlled redundancy to detect and correct errors
- ECC depends on the system requirements and the nature of the channel
- The key in ECC is to find a way to add redundancy to the channel so that the receiver can fully utilize that redundancy to detect and correct the errors, and to reduce the required transmit power – coding gain

Example

- We want to transmit data over a telephone link using a modem under the following conditions
  - Link bandwidth = 3kHz
  - The modem can operate up to the speed of 3600 bits/sec at an error probability $P_e = 8 \times 10^{-4}$

- Target: transmit the data at rate of 1200 bits/sec at maximum output SNR = 13 dB with a prob. of error 10^{-4}

Solution: Shannon Theorem

- Channel capacity is

$$C = B\log_2\left(1+\frac{S}{N}\right) = 13,000 \text{ bits/sec}$$

  Since B = 3000 and S/N = 20 (13 dB = $10\log_{10}20$)
- Thus, by Shannon's theorem, we can transmit the data with an arbitrarily small error probability
- Note that without coding $P_e = 8\times 10^{-4}$

  For the given modem, criterion $P_e = 10^{-4}$ is <u>not met.</u>

Solution: A Simple Code Design

- Repetition code: every bit is transmitted 3 times

  when b_k = "0" or "1", transmit codeword "000" or "111"

- Based on the received codewords, the decoder attempts to extract the transmitted bits using majority-logic decoding scheme
- Clearly, the transmitted bits will be recovered correctly as long as no more than one of the bits in the codeword is affected by noise

Tx bits b_k	0	0	0	0	1	1	1	1
Codewords	000	000	000	000	111	111	111	111
Rx bits	000	001	010	100	011	101	110	111
$\widehat{b}_k$	0	0	0	0	1	1	1	1

- With this simple error control coding, the probability of error is

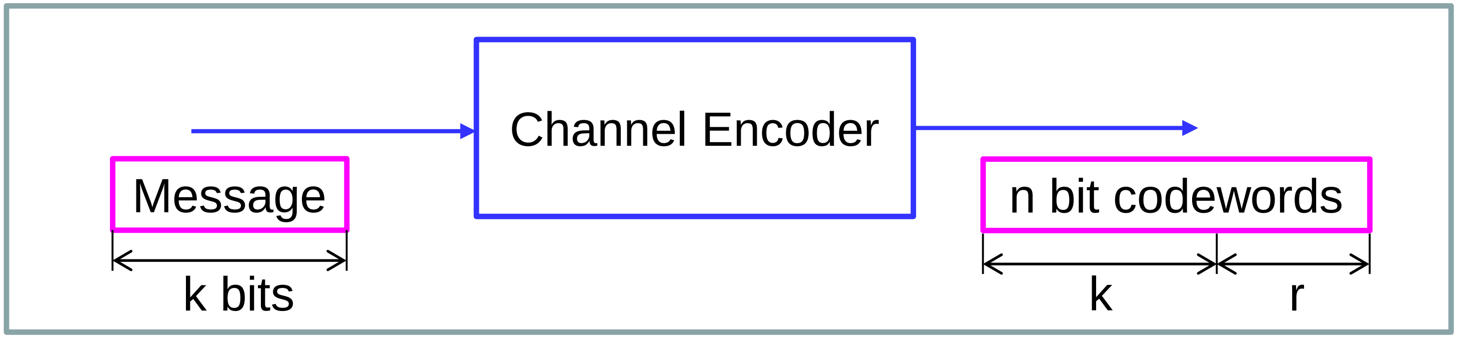
$$
\begin{aligned}
P_e &= P(b_k \neq \hat{b}_k) \\
&= P\ (2 \text{ or more bits in codeword are in error}) \\
&= \binom{3}{2} q_c^2 (1 - q_c) + \binom{3}{3} q_c^3 \\
&= 3q_c^2 - 2q_c^3 \\
&= 0.0192 \times 10^{-4} \\
&\leq \text{Required } P_e \text{ of } 10^{-4}
\end{aligned}
$$

Channel Coding

- Coding techniques are classified as either block codes or convolutional codes, depending on the presence or absence of memory
- A block code has no memory
  - Information sequence is broken into blocks of length k
  - Each block of k infor. bits is encoded into a block of n coded bits
  - No memory from one block to another block
- A convolutional code has memory
  - A shift register of length $k_0 L$ is used.
  - Information bits enter the shift register k_0 bits at a time; then n_0 coded bits are generated
  - These n_0 bits depend not only on the recent k_0 bit that just entered the shift register, but also on the $k_0(L-1)$ previous bits.

Block Codes

- An *(n,k)* block code is a collection of $M = 2^k$ codewords of length n
- Each codeword has a block of k information bits followed by a group of $r = n\text{-}k$ check bits that are derived from the k information bits in the block preceding the check bits



- The code is said to be linear if any linear combination of 2 codewords is also a codeword
  - i.e. if c_i and c_j are codewords, then $c_i + c_j$ is also a codeword (where the addition is always module-2)

- Code rate (rate efficiency) = $\frac{k}{n}$
- Matrix description
  - codeword $\mathbf{c} = (c_1, c_2, ..., c_n)$
  - message bits $\mathbf{m} = (m_1, m_2, ..., m_k)$
- Each block code can be generated using a Generator Matrix $\mathbf{G}$ (dim: $k \times n$)
- Given $\mathbf{G}$, then

$$\boxed{\mathbf{c} = \mathbf{m}\mathbf{G}}$$

Codeword message

Generator Matrix G

$$\mathbf{G} = [\mathbf{I}_k | \mathbf{P}]_{k \times n}$$

$$= \left[\boxed{\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix}} \; \boxed{\begin{matrix} p_{11} & p_{12} & \cdots & p_{1,n-k} \\ p_{21} & p_{22} & & p_{2,n-k} \\ \vdots & & \ddots & \vdots \\ p_{k,1} & p_{k,2} & \cdots & p_{k,n-k} \end{matrix}}\right]$$

- $\mathbf{I}_k$ is identity matrix of order k
- $\mathbf{P}$ is matrix of order $k \times (n-k)$, which is selected so that the code will have certain desirable properties

Systematic Codes

- The form of G implies that the 1^{st} k components of any codeword are precisely the information symbols
- This form of linear encoding is called systematic encoding
- Systematic-form codes allow easy implementation and quick look-up features for decoding
- For linear codes, any code is equivalent to a code in systematic form (given the same performance). Thus we can restrict our study to only systematic codes

Example: Hamming Code

- A family of (n,k) linear block codes that have the following parameters:
  - Codeword length $n = 2^m - 1, \quad m \geq 3$
  - # of message bits $k = 2^m - m - 1$
  - # of parity check bits $n - k = m$
  - Capable of providing single-error correction capability with $d_{\min} = 3$

(7, 4) Hamming Code

- Consider a (7,4) Hamming code with generator matrix

$$\mathbf{G} = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

- Find all codewords

Solution

- Let $\mathbf{m} = [1\ \ 1\ \ 1\ \ 1]$

$$\mathbf{c} = \mathbf{mG} = [1\ \ 1\ \ 1\ \ 1]\left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$= [1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1]$$

List of all Codewords

- n = 7, k = 4 $\rightarrow 2^k = 16$ message blocks

Message				codeword						
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	1	0	1
0	0	1	0	0	0	1	0	1	1	1
0	0	1	1	0	0	1	1	0	1	0
0	1	0	0	0	1	0	0	0	1	1
0	1	0	1	0	1	0	1	1	1	0
0	1	1	0	0	1	1	0	1	0	0
0	1	1	1	0	1	1	1	0	0	1
1	0	0	0	1	0	0	0	1	1	0
1	0	0	1	1	0	0	1	0	1	1
1	0	1	0	1	0	1	0	0	0	1
1	0	1	1	1	0	1	1	1	0	0
1	1	0	0	1	1	0	0	1	0	1
1	1	0	1	1	1	0	1	0	0	0
1	1	1	0	1	1	1	0	0	1	0
1	1	1	1	1	1	1	1	1	1	1

Parity Check Matrix

- For each $\mathbf{G}$, it is possible to find a corresponding parity check matrix $\mathbf{H}$

$$\mathbf{H} = \left[\mathbf{P}^T \; |\mathbf{I}_{n-k}\right]_{(n-k)\times n}$$

- $\mathbf{H}$ can be used to verify if a codeword C is generated by G
- Let C be a codeword generated by $\mathbf{G} = [\mathbf{I}_k|\mathbf{P}]_{k\times n}$

$$\Rightarrow \quad \boxed{\mathbf{c}\mathbf{H}^T = \mathbf{m}\mathbf{G}\mathbf{H}^T = 0}$$

Example: Find the parity check matrix of (7,4) Hamming code

Error Syndrome

- Received codeword $\mathbf{r} = \mathbf{c} + \mathbf{e}$

  where $\mathbf{e}$ = Error vector or Error Pattern

  it is 1 in every position where data word is in error

- Example

$$\mathbf{c} = [1 \ \ 0 \ \ 1 \ \ 0]$$

$$\mathbf{r} = [1 \ \ 1 \ \ 0 \ \ 0]$$

$$\mathbf{e} = [0 \ \ 1 \ \ 1 \ \ 0]$$

Error Syndrome (cont'd)

- $\mathbf{s} \triangleq \mathbf{r}\mathbf{H}^T$ = Error Syndrome
- But

$$\mathbf{s} = \mathbf{r}\mathbf{H}^T = (\mathbf{c}+\mathbf{e})\mathbf{H}^T$$
$$= \mathbf{c}\mathbf{H}^T + \mathbf{e}\mathbf{H}^T$$
$$= \mathbf{e}\mathbf{H}^T$$

1. If s=0 $\rightarrow$ r = c and $\mathbf{m}$ is the 1st k bits of *r*
2. If s ≠0, and s is the jth row of $\mathbf{H}^T \rightarrow$ 1 error in jth position of r

- Consider the (7,4) Hamming code example

$$\mathbf{H}^T = \left[\mathbf{P}^T | \mathbf{I}_{n-k}\right]^T = \begin{bmatrix} \mathbf{P} \\ \mathbf{I}_{n-k} \end{bmatrix}$$

$$= \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

How many error syndromes in total

- Note that s is the last row of H^T
- Also note error took place in the last bit

=> Syndrome indicates position of error

- So if $\mathbf{r} = [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]$

$$\Longrightarrow \quad \mathbf{r}\mathbf{H}^T = [0 \;\; 0 \;\; 0]$$

- But if $\mathbf{r} = [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 0]$

$$\Longrightarrow \quad \mathbf{r}\mathbf{H}^T = [0 \;\; 0 \;\; 1]$$

= Error syndrome $\mathbf{s}$

Cyclic Codes

- A code $C = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{2^k}\}$ is cyclic if

$$(c_1, c_2, \ldots, c_n) \in C \Rightarrow (c_n, c_1, \ldots, c_{n-1}) \in C$$

- (7,4) Hamming code is cyclic

message	codeword
0001	0001101
1000	1000110
0100	0100011

Important Parameters

- Hamming Distance between codewords c_i and c_j:

  $d(c_i, c_j)$ = # of components at which the 2 codewords differ

- Hamming weight of a codeword c_i is

  $w(c_i)$ = # of non-zero components in the codeword

- Minimum Hamming Distance of a code:

  $d_{min} = \min d(c_i, c_j)$ for all $i \neq j$

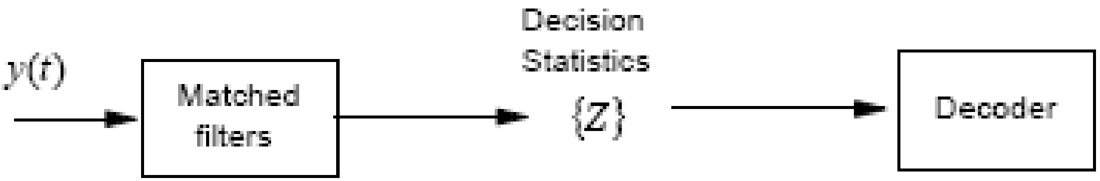
- Minimum Weight of a code:

  $w_{min} = \min w(c_i)$ for all $c_i \neq 0$

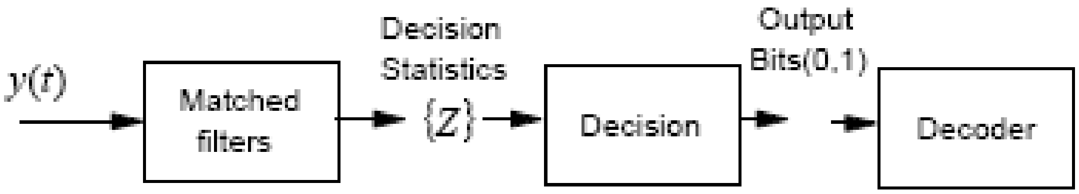
- Theorem: In any linear code, $d_{\min} = w_{\min}$
- Exercise: Find d_{min} for (7,4) Hamming code

Soft-Decision and Hard-Decision Decoding

- **Soft-decision decoder** operates directly on the decision statistics



- **Hard-decision decoder** makes "hard" decision (0 or 1) on individual bits

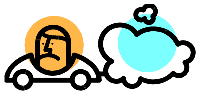


- Here we only focus on hard-decision decoder

Hard-Decision Decoding

- Minimum Hamming Distance Decoding
  - Given the received codeword $\mathbf{r}$, choose c which is closest to r in terms of Hamming distance
  - To do so, one can do an exhaustive search

    – too much if k is large.



- Syndrome Decoding
  - Syndrome testing: $\mathbf{r} = \mathbf{c} + \mathbf{e}$ with $\mathbf{s} = \mathbf{r}\mathbf{H}^T$
  - This implies that the corrupted codeword r and the error pattern have the same syndrome
  - A simplified decoding procedure based on the above observation can be used

Standard Array

- Let the codewords be denoted as $\{c_1, c_2, \ldots, c_M\}$ with c_1 being the all-zero codeword
- A standard array is constructed as

Coset leader

2^k

Syndrome s

$$
2^{n-k}\left\{
\begin{array}{lll}
c_1 & c_2 & c_M \\
e_1 & e_1 \oplus c_2 & e_1 \oplus c_M \\
e_2 & e_2 \oplus c_2 & e_2 \oplus c_M \\
\vdots & \vdots & \vdots \\
e_{2^{n-k}-1} & e_{2^{n-k}-1} \oplus c_2 & e_{2^{n-k}-1} \oplus c_M
\end{array}
\right.
\qquad
\begin{array}{l}
0 \\
s = e_1 H^T \\
s = e_2 H^T \\
\vdots \\
s = e_{2^{n-k}-1} H^T
\end{array}
$$

coset

Error patterns

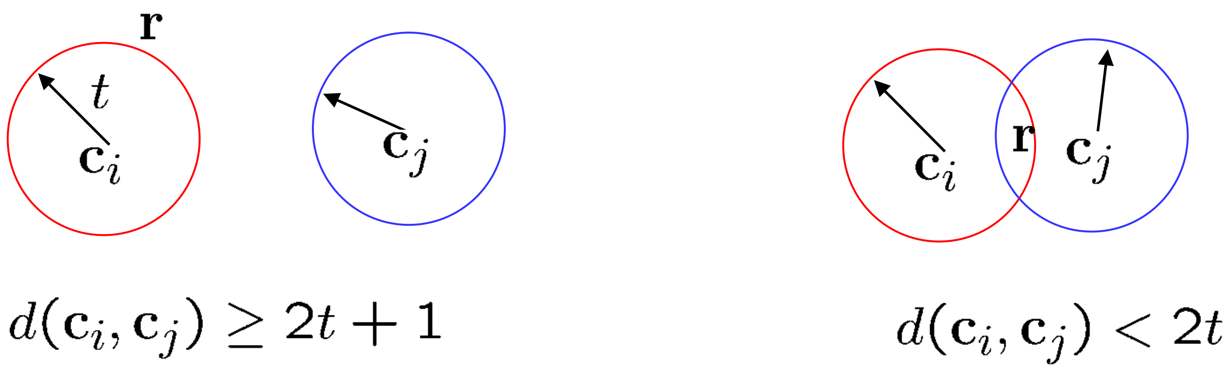
Hard-Decoding Procedure

- Find the syndrome by r using $\mathbf{s}=\mathbf{r}\mathbf{H}^T$
- Find the coset corresponding to s by using the standard array
- Find the cost leader and decode as $\mathbf{c} = \mathbf{r} + \mathbf{e}_j$

- Exercise: try (7,4) Hamming code

Error Correction Capability

- A linear block code with a minimum distance d_{min} can
  - Detect up to $(d_{\min} - 1)$ errors in each codeword
  - Correct up to $t = \lfloor \frac{d_{\min} - 1}{2} \rfloor$ errors in each codeword
  - t is known as the error correction capability of the code



Probability of Codeword Error for Hard-Decision Decoding

- Consider a linear block code (n, k) with an error correcting capability *t*. The decoder can correct all combination of errors up to and including *t* errors.
- Assume that the error probability of each individual coded bit is *p* and that bit errors occur independently since the channel is memoryless
- If we send n-bit block, the probability of receiving a specific pattern of m errors and (n-m) correct bits is

$$p^m(1-p)^{n-m}$$

- Total number of distinct pattern of n bits with m errors and (n-m) correct bits is

$$\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{m!(n-m)!}$$

- Total probability of receiving a pattern with m errors is

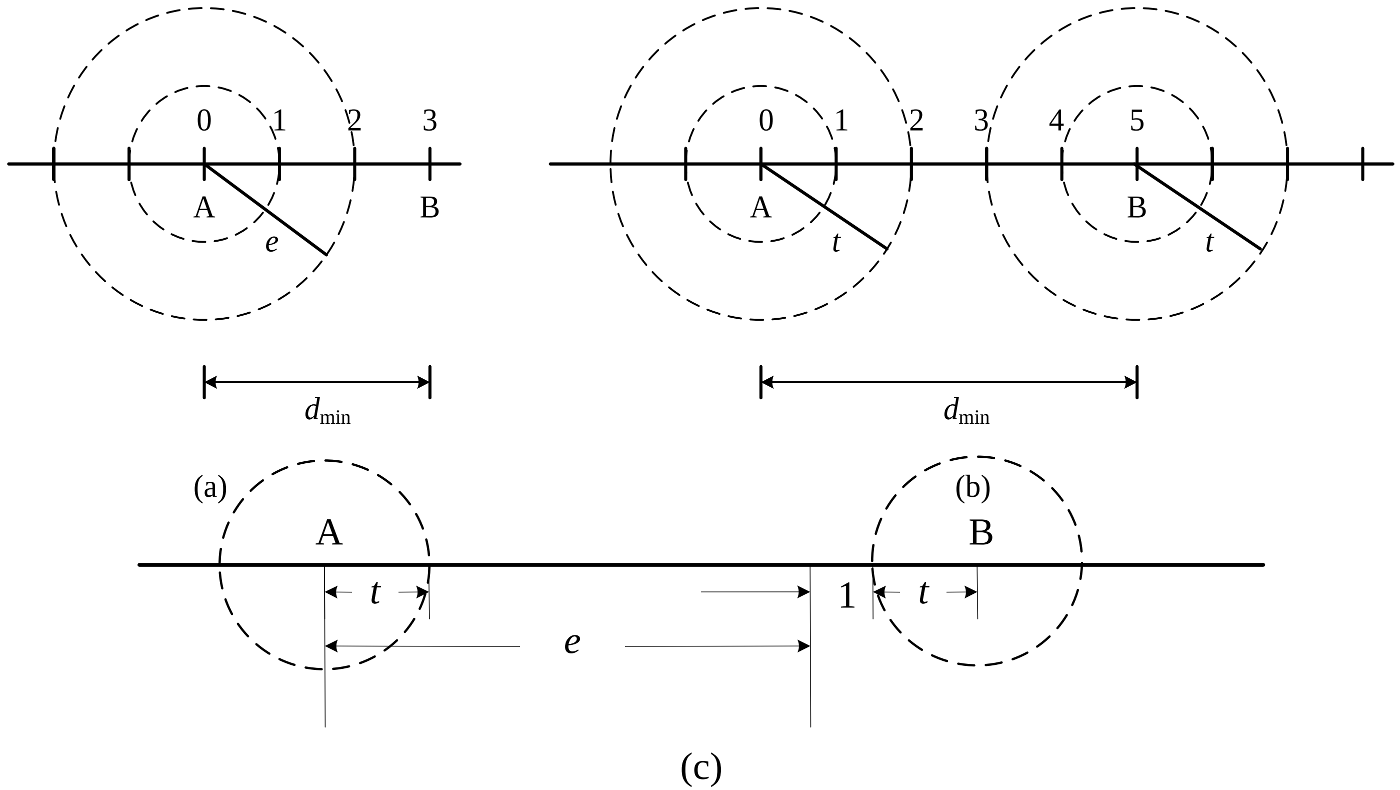
$$P(m,n) = \begin{pmatrix} n \\ m \end{pmatrix} \cdot p^m (1-p)^{n-m}$$

- Thus, the codeword error probability is upper-bounded by

$$P_M \leq \sum_{m=t+1}^{n} \begin{pmatrix} n \\ m \end{pmatrix} p^m (1-p)^{n-m}$$

(with equality for perfect codes)

Error Detection vs. Error Correction



- To detect e bit errors, we have $d_{\min} \geq e+1$
- To correct t bit errors, we have $d_{\min} \geq 2t+1$

Major Classes of Block Codes

- Repetition Code
- Hamming Code
- Golay Code
- BCH Code
- Reed-Solomon Codes
- Walsh Codes
- LDPC Codes: invented by Robert Gallager in his PhD thesis in1960, now proved to be capacity-approaching

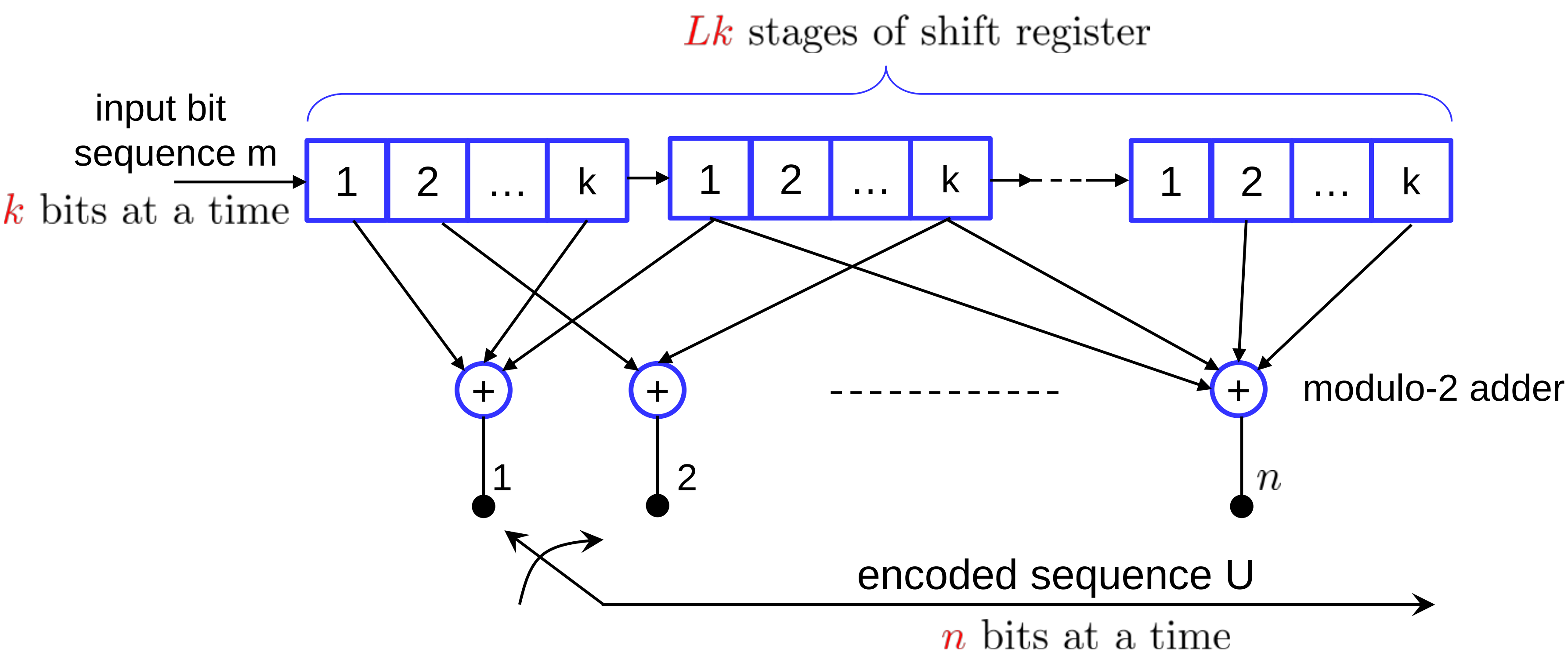
Convolutional Codes

- A convolutional code has memory
  - It is described by 3 integers: n, k, and L
  - Maps k bits into n bits using previous (L-1) k bits
  - The n bits emitted by the encoder are not only a function of the current input k bits, but also a function of the previous (L-1)k bits
  - k/n = Code Rate (information bits/coded bit)
  - L is the constraint length and is a measure of the code memory
  - n does not define a block or codeword length

Convolutional Encoding

- A rate k/n convolutional encoder with constraint length L consists of
  - kL-stage shift register and n mod-2 adders
- At each unit of time:
  - k bits are shifted into the 1st k stages of the register
  - All bits in the register are shifted k stages to the right
  - The output of the n adders are sequentially sampled to give the coded bits
  - There are n coded bits for each input group of k information or message bits. Hence R = k/n information bits/coded bit is the code rate (k<n)

Encoder Structure
(rate k/n, constraint length L)



- Typically, k=1 for binary codes. Hence, consider rate 1/n codes

Convolution Codes Representation

- Encoding function: characterizes the relationship between the information sequence **m** and the output coded sequence **U**
- Four popular methods for representation
  - ✓ Connection pictorial and connection polynomials (usually for encoder)
  - ✓ State diagram
  - Tree diagram
  - ✓ Trellis diagram

Usually for decoder

Connection Representation

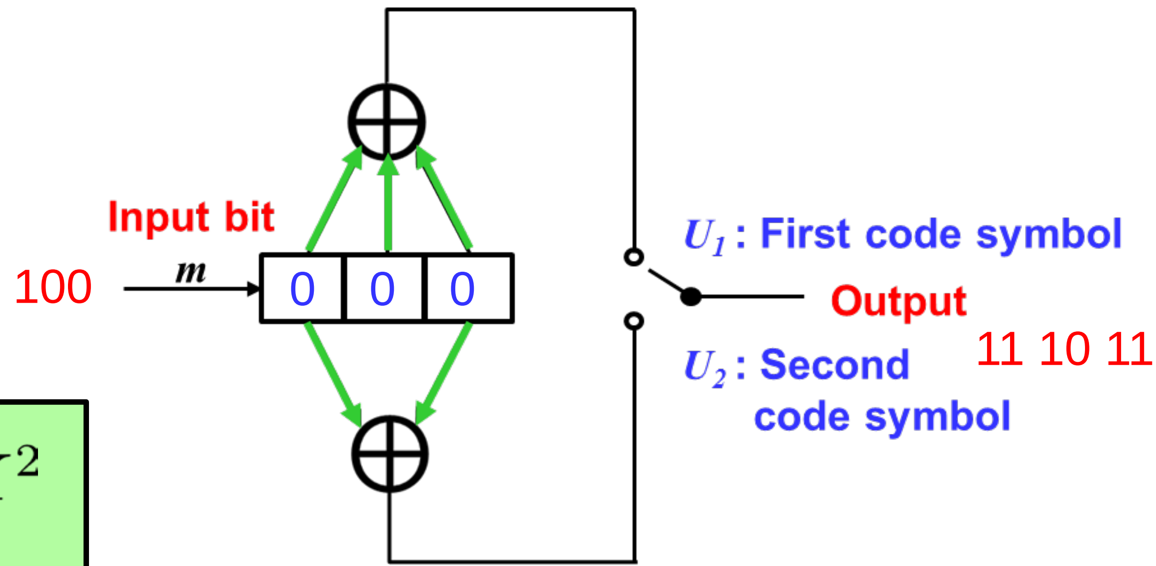
- Specify n connection vectors, $\mathbf{g}_i, i = 1, \ldots, n$ for each of the n mod-2 adders
- Each vector has kL dimension and describes the connection of the shift register to the mod-2 adders
- A 1 in the i^{th} position of the connection vector implies shift register is connected
- A 0 implies no connection exists

Example: L = 3, Rate 1/2

$$g_1 = 1\,1\,1$$
$$g_2 = 1\,0\,1$$

Or

$$g_1(X) = 1 + X + X^2$$
$$g_2(X) = 1 + X^2$$

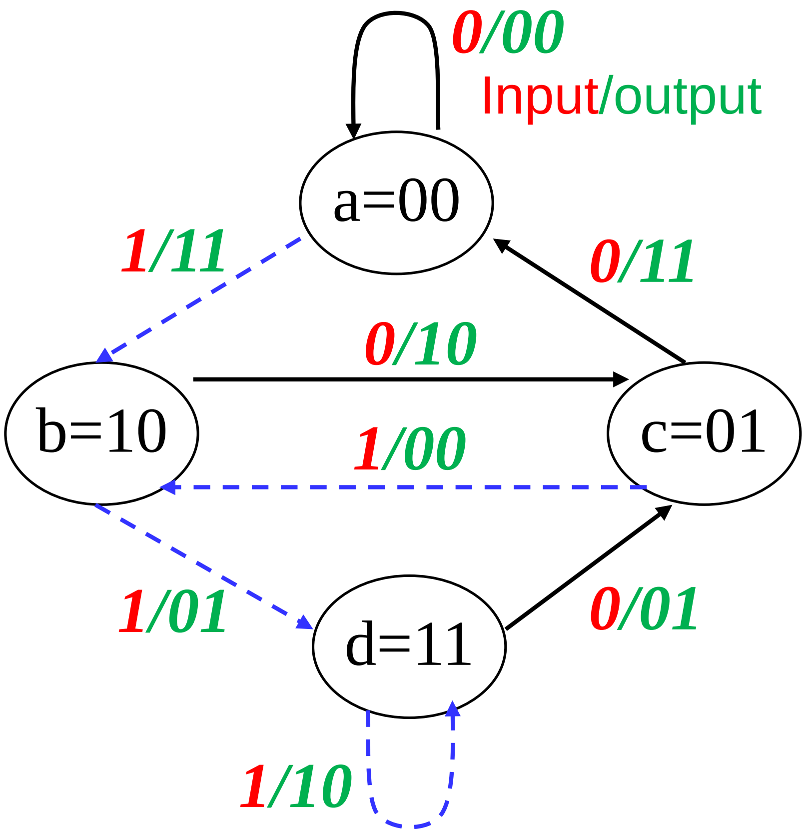


If Initial Register content is 0 0 0 and Input Sequence is 1 0 0. Then Output Sequence is 11 10 11

State Diagram Representation

- The contents of the rightmost L-1 stages (or the previous L-1 bits) are considered the current state => 2^{L-1} states
- Knowledge of the current state and the next input is necessary and sufficient to determine the next output and next state
- For each state, there are only 2 transitions (to the next state) corresponding to the 2 possible input bits
- The transitions are represented by paths on which we write the output word associated with the state transition
  - A solid line path corresponds to an input bit 0
  - A dashed line path corresponds to an input bit 1

Example: L =3, Rate = 1/2



Current State	Input	Next State	Output
00	0	00	***00***
	1	10	***11***
10	0	01	***10***
	1	11	***01***
01	0	00	***11***
	1	10	***00***
11	0	01	***01***
	1	11	***10***

Example

- Assume that m =11011 is the input followed by L-1 = 2 zeros to flush the register. Also assume that the initial register contents are all zero. Find the output sequence U

Input bit m_i	Register contents	State at time t_i	State at time t_{i+1}	Branch word at time t_i u_1	u_2
--	000	00	00	--	--
1	100	00	10	1	1
1	110	10	11	1	1
0	011	11	01	0	0
1	101	01	10	1	1
1	110	10	11	1	1
0	011	11	01	0	0
0	001	01	00	0	0

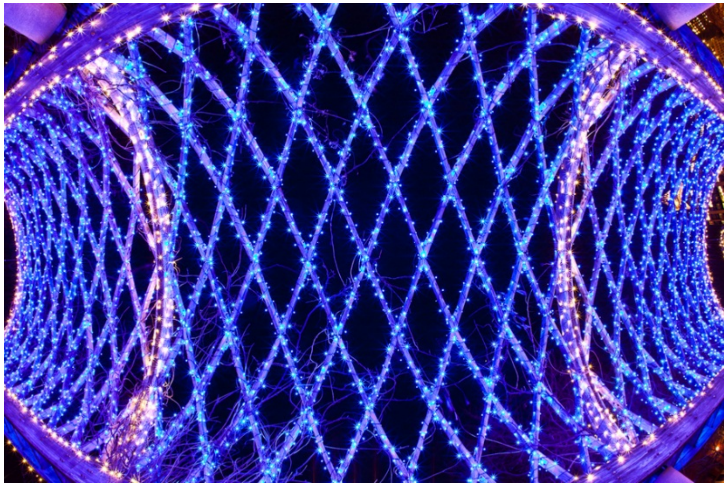
State t_i

State t_{i+1}

Output sequence: **U** = **11 01 01 00 01 01 11**

Trellis Diagram

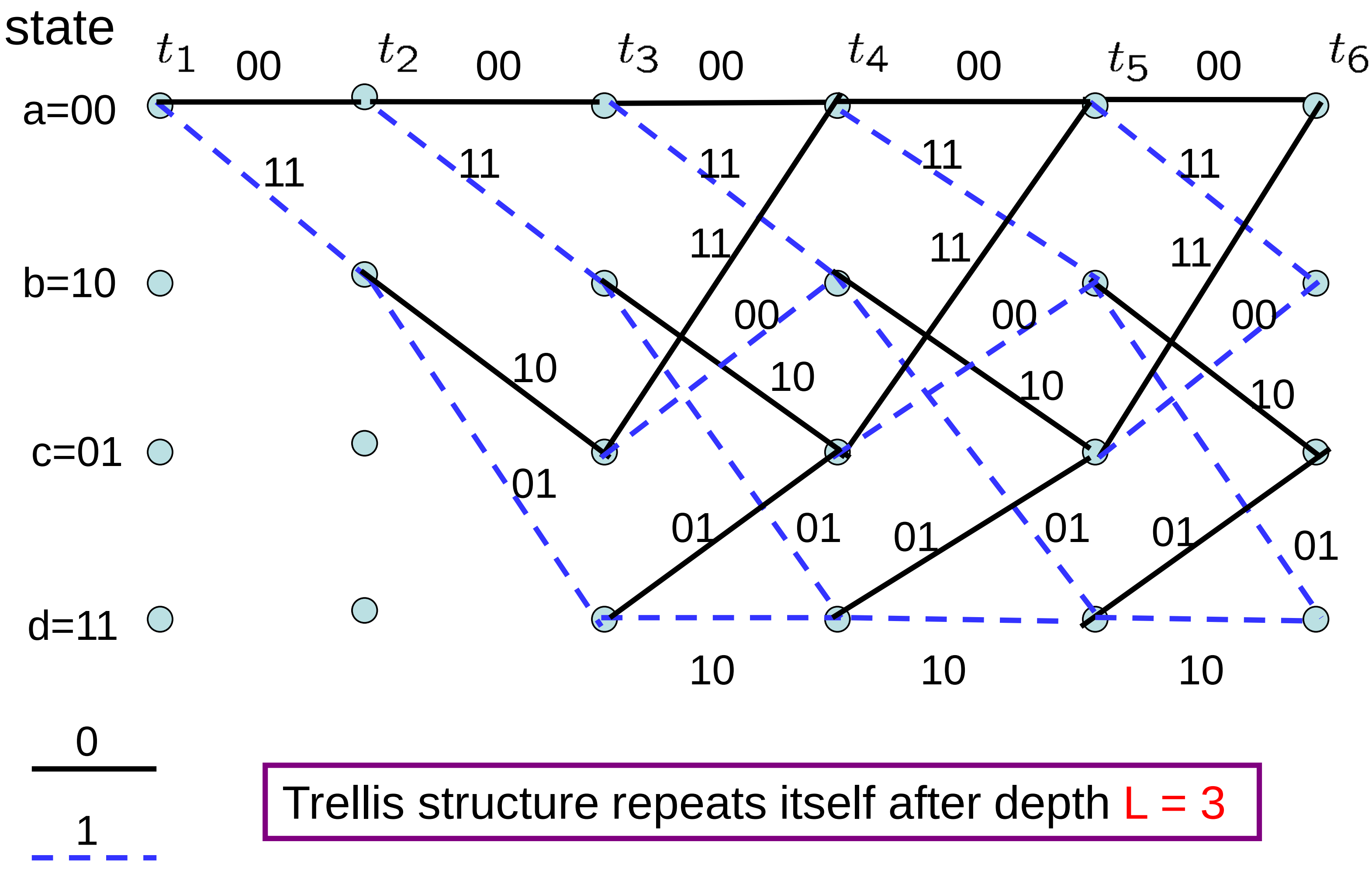
- The trellis diagram is similar to the state diagram, except that it adds the dimension of time
- The code is represented by a trellis where each trellis branch describes an output word



Blue trellis lights

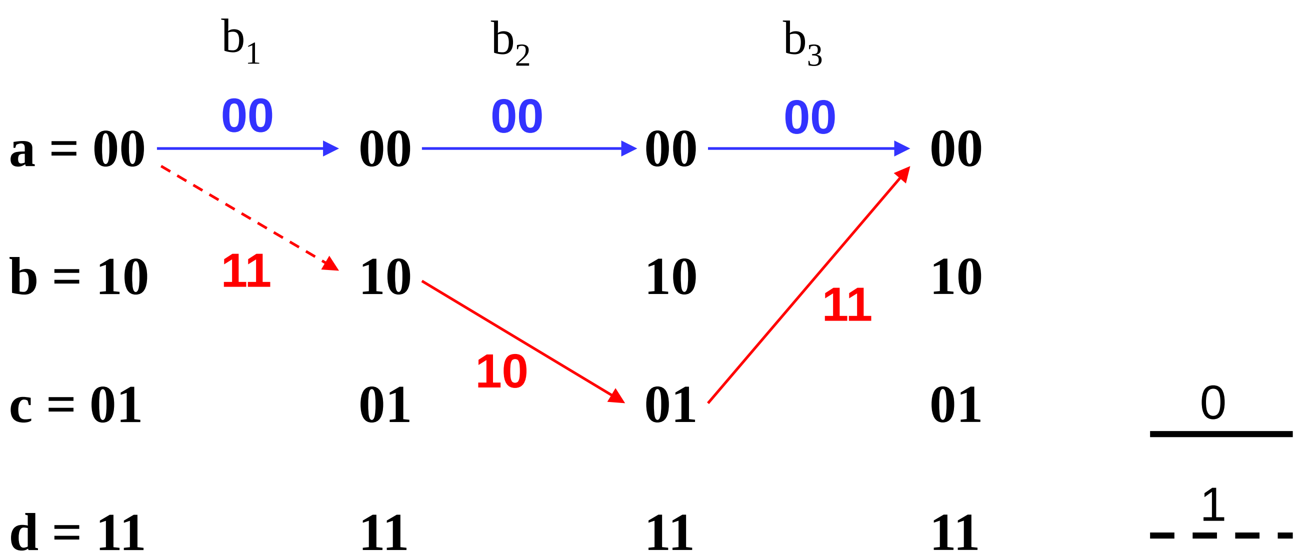
-- Columbus Park, New York City

Trellis Diagram



Trellis structure repeats itself after depth L = 3

- Every input sequence $(m_1, m_2, \ldots)$ corresponds to
  - a path in the trellis
  - a state transition sequence $(s_0, s_1, \ldots)$, (assume $s_0 = \mathbf{0}$ is fixed)
  - an output sequence $((u_1, u_2), (u_3, u_4), \ldots)$
- Example: Let $s_0 = 00$, then
  - $b_1 b_2 b_3 = 000$ gives output *000000* and states *aaaa*
  - $b_1 b_2 b_3 = 100$ gives output *111011* and states *abca*



- We have introduced conv. code
  - Constraint length L and rate R = 1/n
  - Polynomials representation
  - State diagram representation
  - Trellis diagram representation

- We will talk about decoding of convolutional code
  - Maximum Likelihood Decoding
  - Viterbi Algorithm
  - Transfer Function

Maximum Likelihood Decoding

- Transmit a coded sequence $\mathbf{U}^{(\mathbf{m})}$ (correspond to message sequence $\mathbf{m}$) using a digital modulation scheme (e.g. BPSK or QPSK)
- Received sequence $\mathbf{Z}$
- Maximum likelihood decoder
  - Find the sequence $\mathbf{U}^{(\mathbf{j})}$ such that

$$P(\mathbf{Z}|\mathbf{U}^j) = \max_{\forall \mathbf{U}^{(m)}} P(\mathbf{Z}|\mathbf{U}^{(m)})$$

  - Will minimize the probability of error if $\mathbf{m}$ is equally likely

Maximum Likelihood Metric

- Assume a memoryless channel, i.e. noise components are independent. Then, for a rate 1/n code

$$P(\mathbf{Z}|\mathbf{U}^{(\mathbf{m})}) = \prod_{i=1}^{\infty} P(Z_i|U_i^{(m)}) = \prod_{i=1}^{\infty}\prod_{j=1}^{n} P(z_{ji}|u_{ji}^{(m)})$$

i-th branch of **Z**

- Then the problem is to find a path through the trellis such that

$$\max_{\mathbf{U}^{(m)}} \prod_{i=1}^{\infty}\prod_{j=1}^{n} P(z_{ji}|u_{ji}^{(m)})$$

by taking log

Log-likelihood path metric

$$\max_{\mathbf{U}^{(m)}} \boxed{\sum_{i=1}^{\infty}\sum_{j=1}^{n} \log P(z_{ji}|u_{ji}^{(m)})}$$

i-th branch metric

$$= \max_{\mathbf{U}^{(m)}} \sum_{i=1}^{\infty}\sum_{j=1}^{n} LL\left(z_{ji}|u_{ji}^{(m)}\right)$$

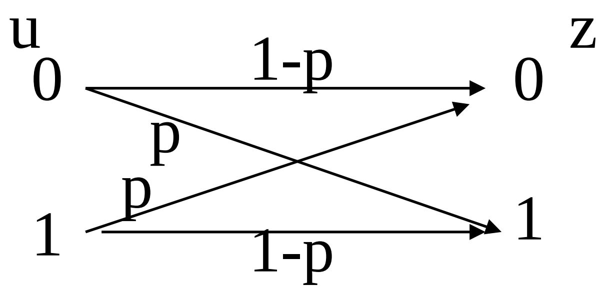
Log-likelihood of $z_{ji}|u_{ji}^{(m)}$

Decoding Algorithm: Log-Likelihood

- For AWGN channel (soft-decision)
  - $z_{ji} = u_{ji} + n_{ji}$ and P($z_{ji}|u_{ji}$) is Gaussian with mean u_{ji} and variance σ^2
  - Hence
$$\ln p(z_{ji}|u_{ji}) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(z_{ji}-u_{ji})^2}{2\sigma^2}$$
  - Note that the objective is to compare which Σ_i ln(p(z|u)) for different **u** is larger, hence, constant and scaling does not affect the results
  - Then, we let the log-likelihood be $LL(z_{ji}|u_{ji}) = -(z_{ji}-u_{ji})^2$

  and
$$\boxed{\log P(Z|U^{(m)}) = -\sum_{i=1}^{\infty}\sum_{j=1}^{n}\left(z_{ji}-u_{ji}^{(m)}\right)^2}$$
  - Thus, soft decision ML decoder is to choose the path whose corresponding sequence is at the minimum Euclidean distance to the received sequence

- For binary symmetric channel (hard decision)



$$p(z|u) = \begin{cases} p & \text{if } z \neq u \\ 1-p & \text{if } z = u \end{cases}$$

$$LL(z_{ji}|u_{ji}) = \ln p(z_{ji}|u_{ji}) = \begin{cases} \ln p & \text{if } z_{ji} \neq u_{ji_i} \\ \ln(1-p) & \text{if } z_{ji} = u_{ji} \end{cases}$$

$$= \begin{cases} \ln p/(1-p) & \text{if } z_{ji} \neq u_{ji} \\ 0 & \text{if } z_{ji} = u_{ji} \end{cases}$$

$$= \begin{cases} -1 & \text{if } z_{ji} \neq u_{ji} \\ 0 & \text{if } z_{ji} = u_{ji} \end{cases}$$ (since $p<0.5$)

- Thus

$$\boxed{\log P(Z|U^{(m)}) = -d_m}$$

Hamming distance between Z and $U^{(m)}$, i.e. they differ in d_m positions

Hard-Decision ML Decoder = Minimum Hamming Distance Decoder

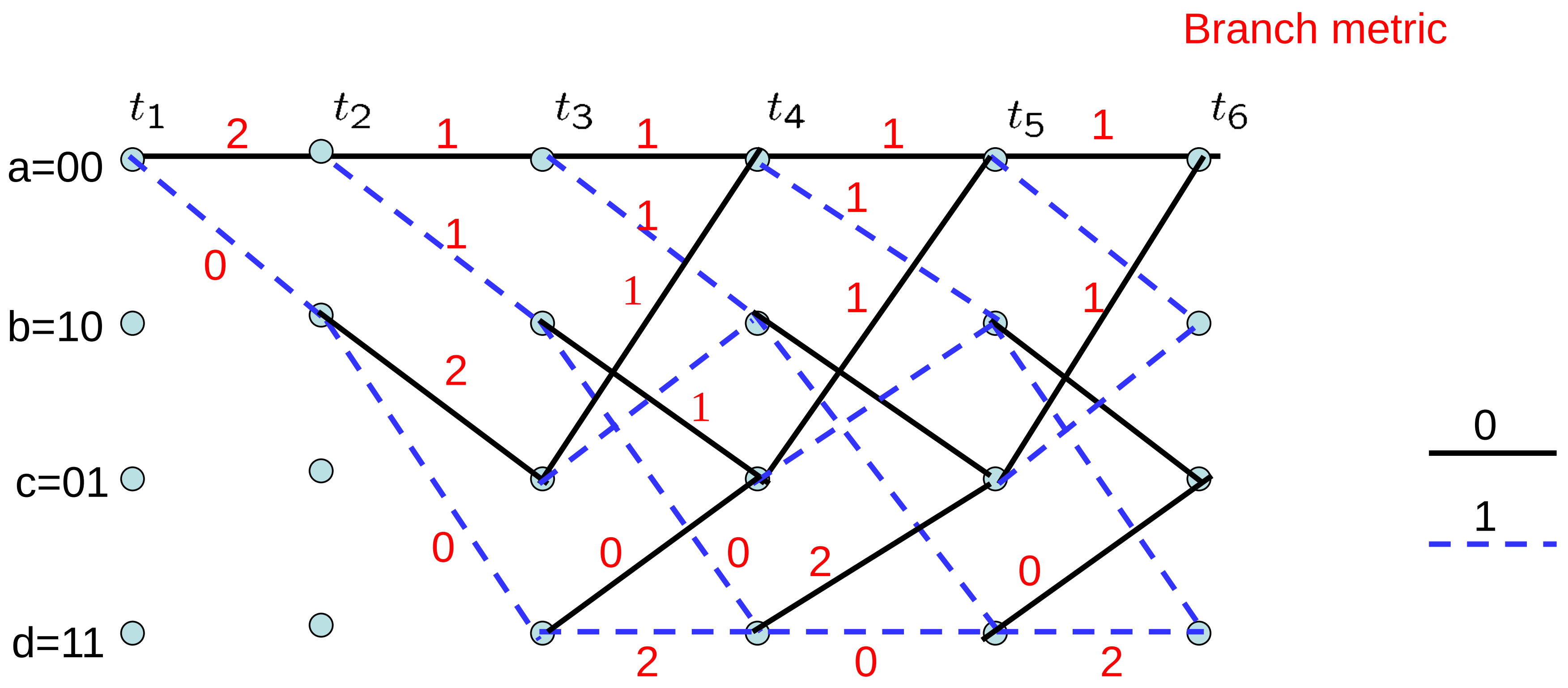
- Maximum Likelihood Decoding Procedure
  - Compute, for each branch i, the branch metric using the output bits $\{u_{1,i}, u_{2,i}, \ldots, u_{n,i}\}$ associated with that branch and the received symbols $\{z_{1,i}, z_{2,i}, \ldots, z_{n,i}\}$
  - Compute, for each valid path through the trellis (a valid codeword sequence $\mathbf{U}^{(m)}$), the sum of the branch metrics along that path
  - The path with the maximum path metric is the decoded path
- To compare all possible valid paths we need to do exhaustive search or brute-force, not practical as the # of paths grow exponentially as the path length increases
- The optimum algorithm for solving this problem is the **Viterbi decoding algorithm** or **Viterbi decoder**

- BS & MS in MIT
- PhD in University of Southern California
- Invention of Viterbi algorithm in 1967
- Co-founder of Qualcomm Inc. in 1983

Andrew Viterbi
(1935-)

Viterbi Decoding (R=1/2, L=3)

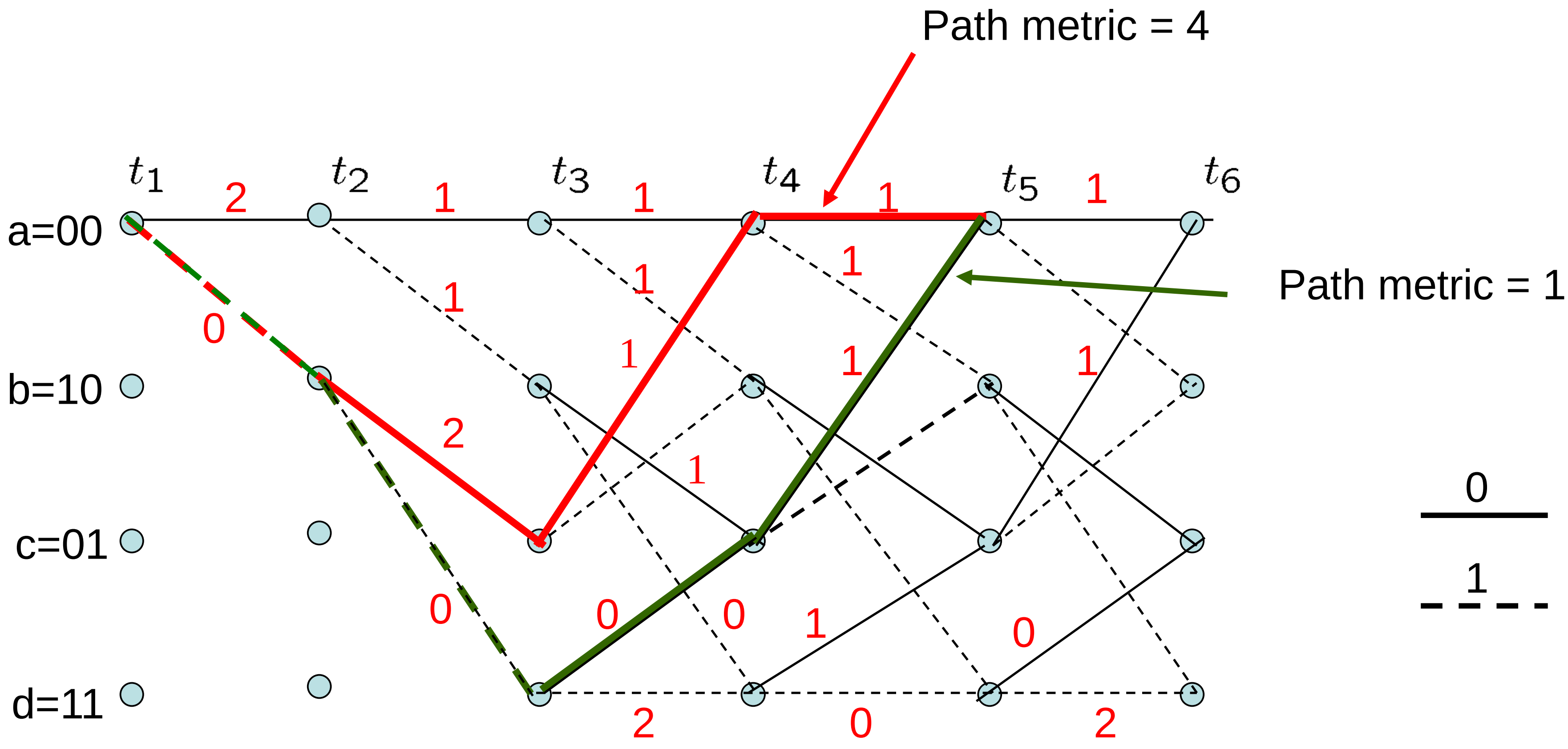
Input data sequence **m**:	1	1	0	1	1	…
Coded sequence **U**:	11	0 1	01	00	01	…
Received sequence **Z**:	11	0 1	01	10	01	…



Viterbi Decoder

- Basic idea:
  - If any 2 paths in the trellis merge to a single state, one of them can always be eliminated in the search
- Let cumulative path metric of a given path at t_i = sum of the branch metrics along that path up to time t_i
- Consider t_5
  - The upper path metric is 4, the lower math metric is 1
  - The upper path metric CANNOT be part of the optimum path since the lower path has a lower metric
  - This is because future output branches depend only on the current state and not the previous state

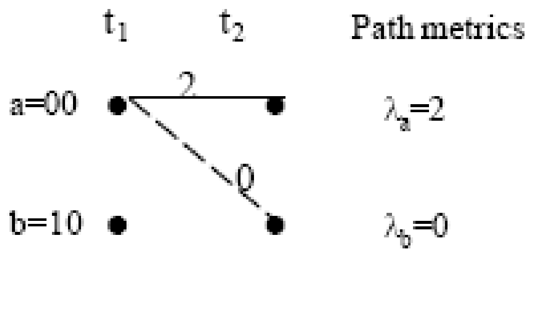
Path Metrics for 2 Merging Paths



Viterbi Decoding

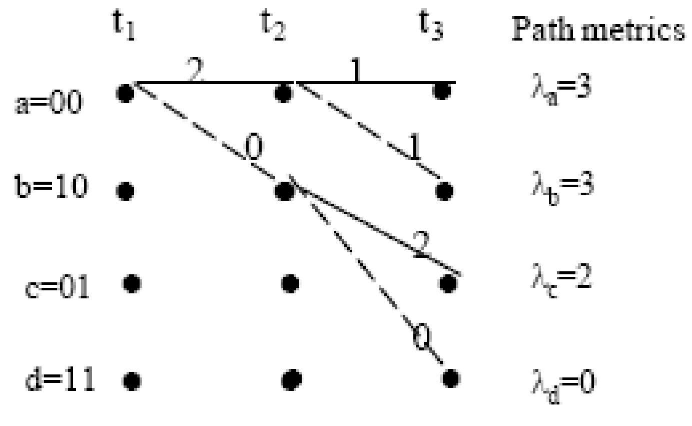
- At time t_i, there are 2^{L-1} states in the trellis
- Each state can be entered by means of 2 states
- Viterbi Decoding consists of computing the metrics for the 2 paths entering each state and eliminating one of them
- This is done for each of the 2^{L-1} nodes at time t_i
- The decoder then moves to time t_{i+1} and repeats the process

Example



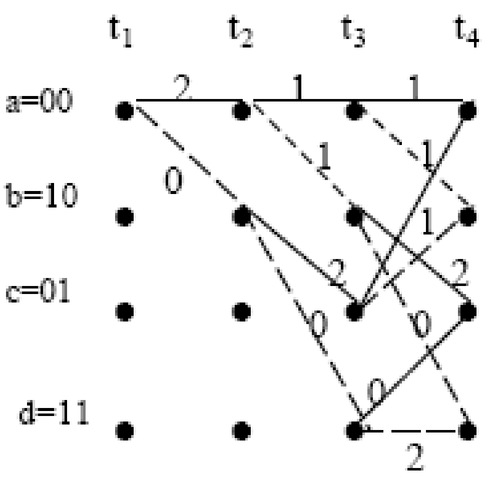
t_1
t_2
Path metrics
a=00
b=10
2
0
$\lambda_a=2$
$\lambda_b=0$

(a)

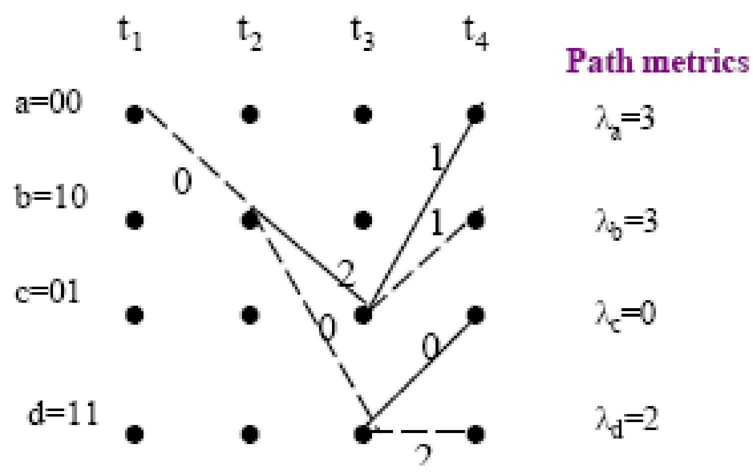


t_1
t_2
t_3
Path metrics
a=00
b=10
c=01
d=11
2
1
0
1
2
0
$\lambda_a=3$
$\lambda_b=3$
$\lambda_c=2$
$\lambda_d=0$

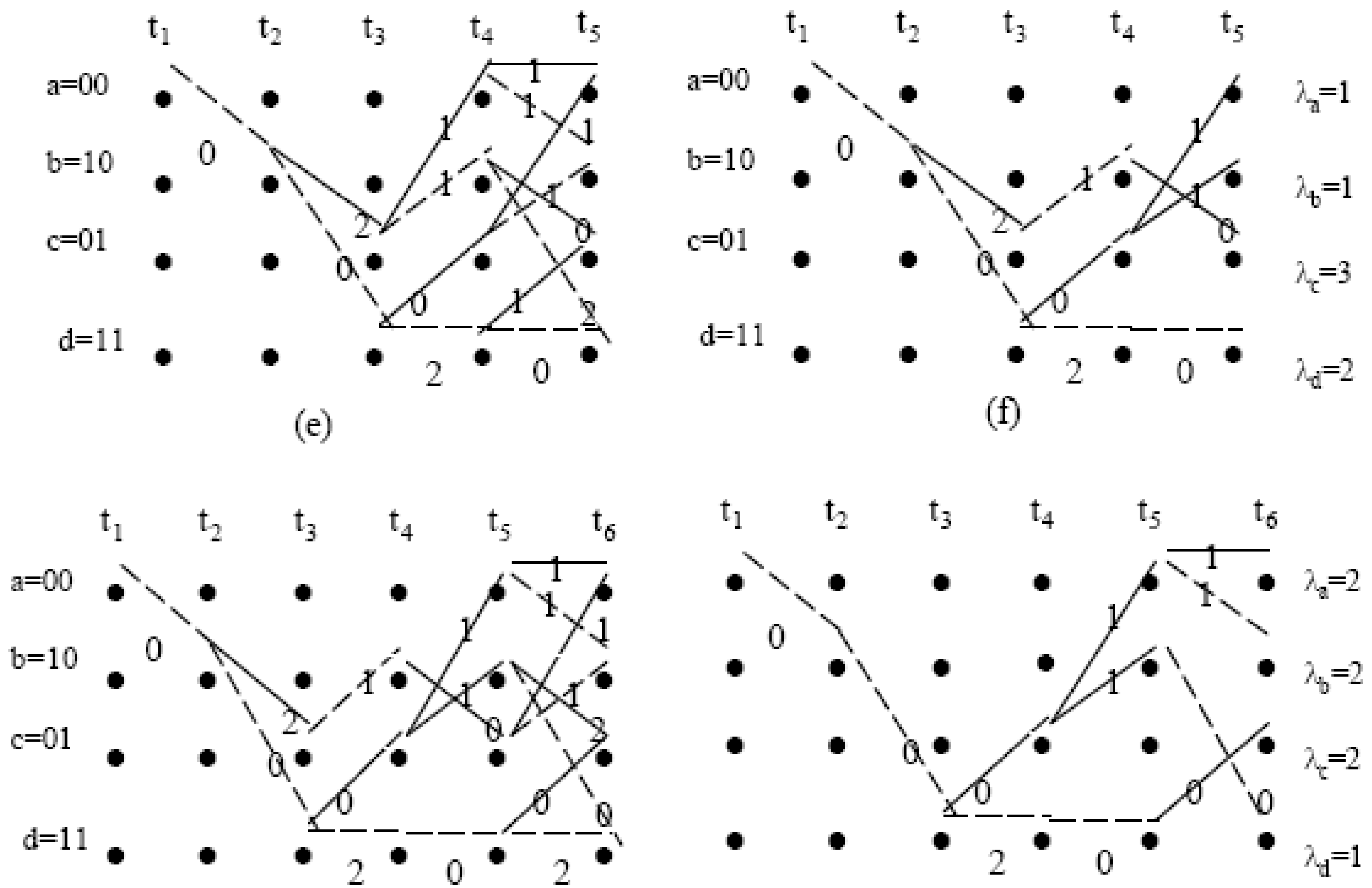
(b)



t_1
t_2
t_3
t_4
a=00
b=10
c=01
d=11
2
1
1
0
1
1
1
2
2
0
0
0
2



t_1
t_2
t_3
t_4
Path metrics
a=00
b=10
c=01
d=11
0
2
0
1
1
0
2
$\lambda_a=3$
$\lambda_b=3$
$\lambda_c=0$
$\lambda_d=2$

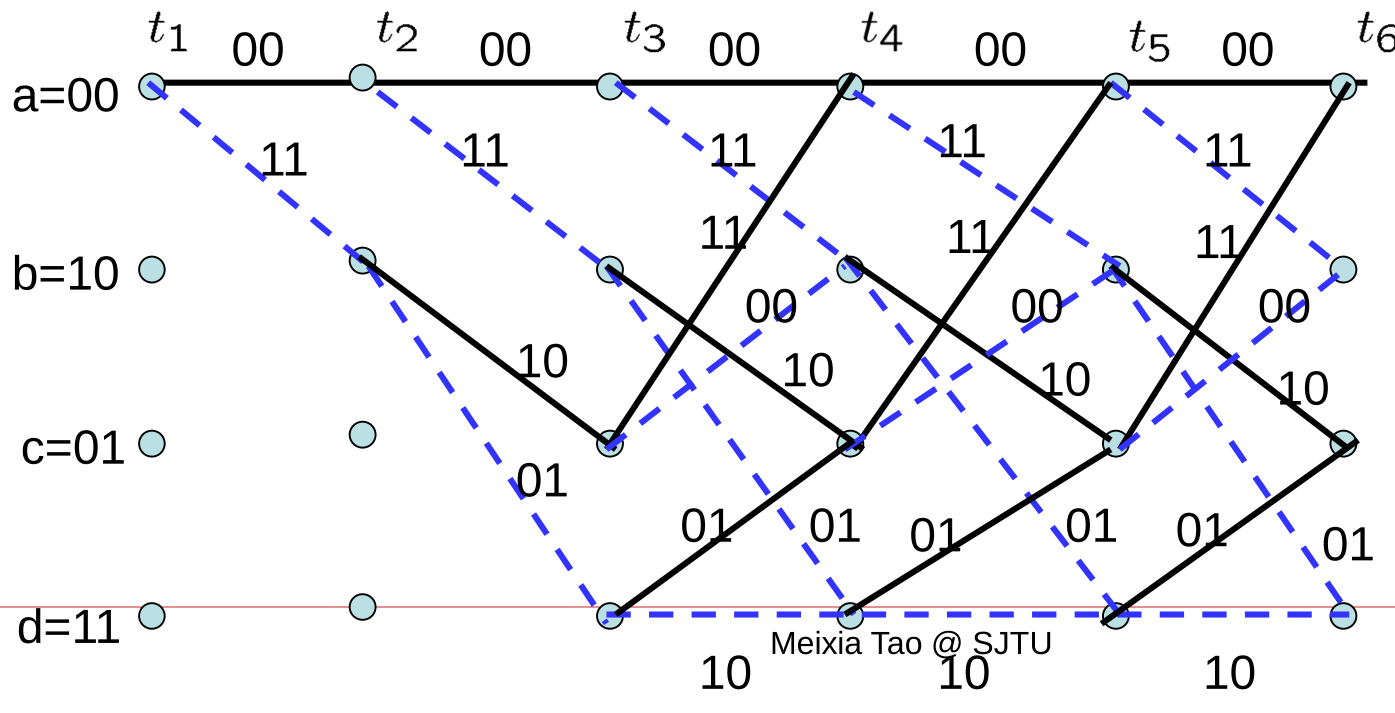


t1 t2 t3 t4 t5
a=00
b=10
c=01
d=11
(e)
(f)
λa=1
λb=1
λc=3
λd=2
t1 t2 t3 t4 t5 t6
λa=2
λb=2
λc=2
λd=1

Distance Properties

- d_{free}= Minimum Free distance = Minimum distance of any pair of arbitrarily long paths that diverge and remerge
- A code can correct any t channel errors where (this is an approximation)

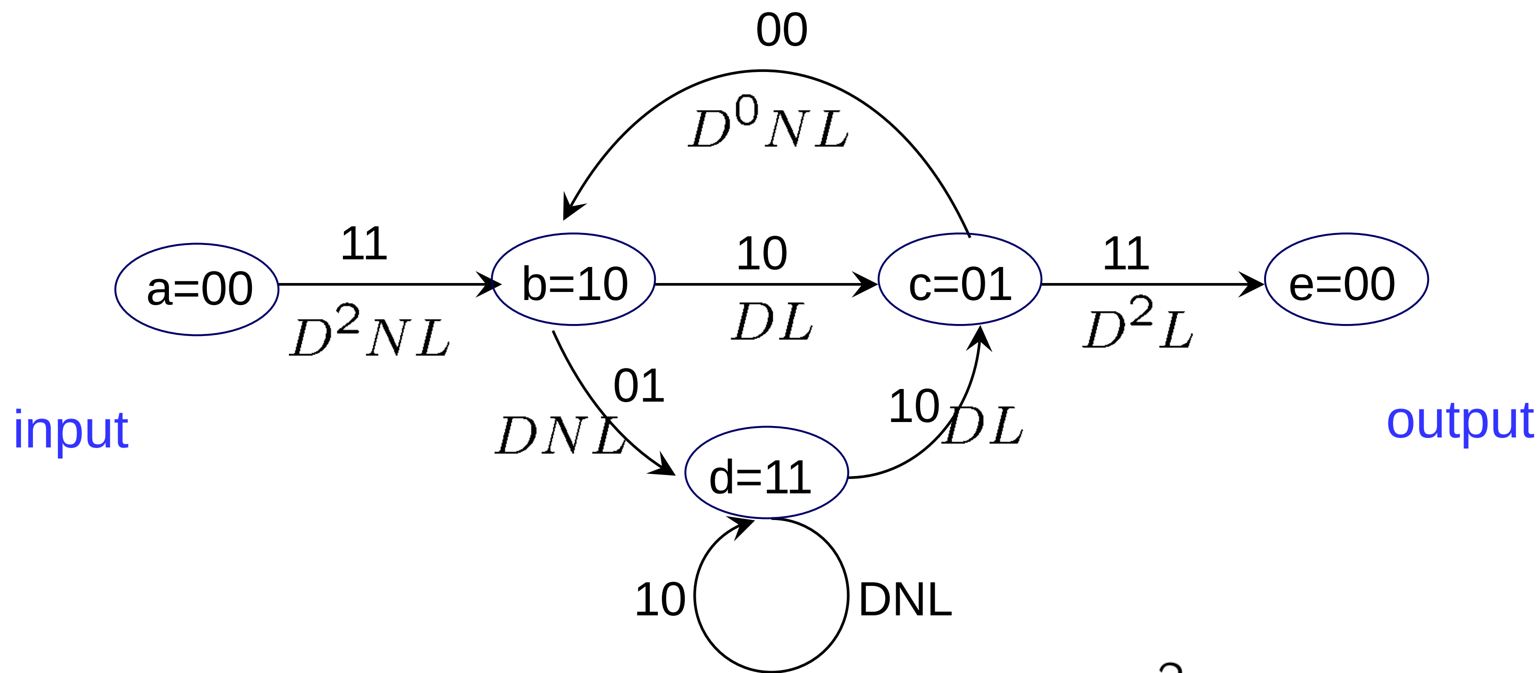
$$t \leq \lfloor \frac{d_{\text{free}} - 1}{2} \rfloor$$



Transfer Function

- The distance properties and the error rate performance of a convolutional code can be obtained from its transfer function
- Since a convolutional code is linear, the set of Hamming distances of the code sequences generated up to some stages in the trellis, from the all-zero code sequence, is the same as the set of distances of the code sequences with respect to any other code sequence
- Thus, we assume that the all-zero path is the input to the encoder

State Diagram Labeled according to distance from all-zero path



- D^m denote m non-zero output bits
- N if the input bit is non-zero
- L denote a branch in the path

$$\begin{cases} X_b = D^2NLX_a + LNX_c \\ X_c = DLX_b + DLX_d \\ X_d = DNLX_b + DNLX_d \\ X_e = D^2LX_c \end{cases}$$

- The transfer function T(D,N,L), also called the weight enumerating function of the code is

$$T(D,N,L)=\frac{X_e}{X_a}$$

- By solving the state equations we get

$$T(D,N,L)=\frac{D^5NL^3}{1-DNL(1+L)}$$
$$=D^5NL^3+D^6N^2L^4(1+L)+D^7N^3L^5(1+L)^2$$
$$+\ldots+D^{l+5}N^{l+1}L^{l+3}(1+L)^l+\ldots$$

- The transfer function indicates that:
  - There is one path at distance 5 and length 3, which differs in 1 input bit from the correct all-zeros path
  - There are 2 paths at distance 6, one of which is of length 4, the other length 5, and both differ in 2 input bits from all-zero path
  - $d_{\text{free}}=5$

Known Good Convolutional Codes

- Good convolutional codes can only be found in general by computer search
- There are listed in tables and classified by their constraint length, code rate, and their generator polynomials or vectors (typically using octal notation).
- The error-correction capability of a convolutional code increases as n increases or as the code rate decreases.
- Thus, the channel bandwidth and decoder complexity increases

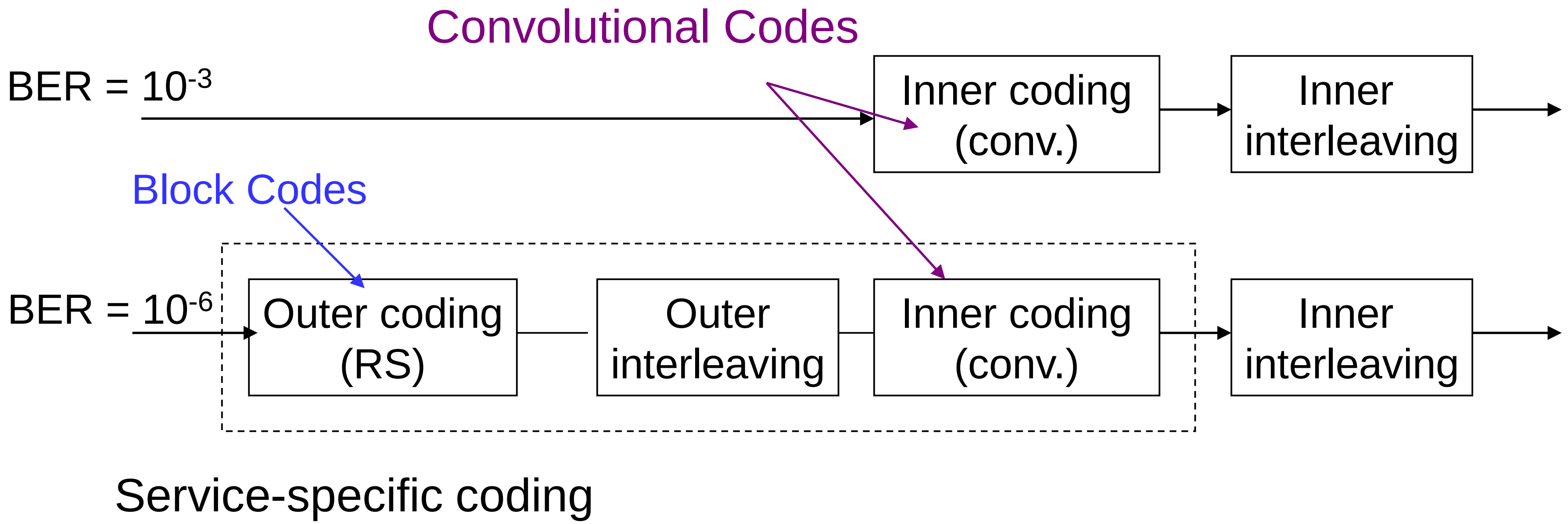
Good Codes with Rate 1/2

Constraint Length	Generator Polynomials	d_{free}
3	(5,7)	5
4	(15,17)	6
5	(23,35)	7
6	(53,75)	8
7	(133,171)	10
8	(247,371)	10
9	(561,753)	12
10	(1167,1545)	12

Good Codes with Rate 1/3

Constraint Length	Generator Polynomials	d_{free}
3	(5,7,7)	8
4	(13,15,17)	10
5	(25,33,37)	12
6	(47,53,75)	13
7	(133,145,175)	15
8	(225,331,367)	16
9	(557,663,711)	18
10	(1117,1365,1633)	20

Basic Channel Coding for Wideband CDMA



Convolutional code is rate 1/3 and rate ½, all with constraint length 9

Channel Coding for Wireless LAN (IEEE802.11a)

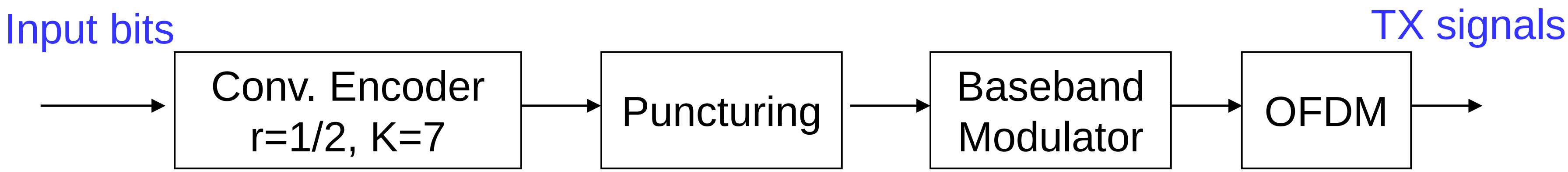


Table 11-3. Encoding details for different OFDM data rates

Speed (Mbps)	Modulation and coding rate (R)	Coded bits per carrier[a]	Coded bits per symbol	Data bits per symbol[b]
6	BPSK, R=1/2	1	48	24
9	BPSK, R=3/4	1	48	36
12	QPSK, R=1/2	2	96	48
18	QPSK, R=3/4	2	96	72
24	16-QAM, R=1/2	4	192	96
36	16-QAM, R=3/4	4	192	144
48	64-QAM, R=2/3	6	288	192
54	64-QAM, R=3/4	6	288	216

Source: 802.11 Wireless Networks: The Definitive Guide / by M. Gast / O'Reilly

Other Advanced Channel Coding

- Low-density parity check codes: Robert Gallager 1960
- Turbo codes: Berrou et al 1993
- Trellis-coded modulation: Ungerboeck 1982
- Space-time coding: Vahid Tarokh et al 1998
- Polar codes: Erdal Arikan 2009

Exercise

- Find out the coding techniques adopted in LTE