1. EQUATIONS

Objectives

To understand that simple equations are a way of finding out an unknown quantity.

To be able to recognise linear equations, quadratics and polynomials and to know that if they are the product of linear factors then they can be solved.

To be able to perform basic algebraic manipulations of equations and to be able to solve them by factorisation or by the formula for quadratics.

To understand the use of the remainder theorem and the method of completing the square.

To understand the nature of equations by investigating their roots.

1.1 Simple Equations

Numbers

To begin with we need to classify numbers. They can be one of three types.

- Integers whole numbers such as 1, 2, -5, 0, etc.
- Rationals fractions such as 1/2, 2/3, -5/17 etc.
- Irrationals values such as $\sqrt{2}$, $\sqrt{5}$, $2\sqrt{3}$ etc.

We can represent these numbers by letters, usually x, y, z, when we do not know their actual value. Such numbers are called **variables.** The following statement can be represented by a **linear equation.**

'There are three times as many male engineers as there are female ones.' can be written mathematically as $y = 3x$.

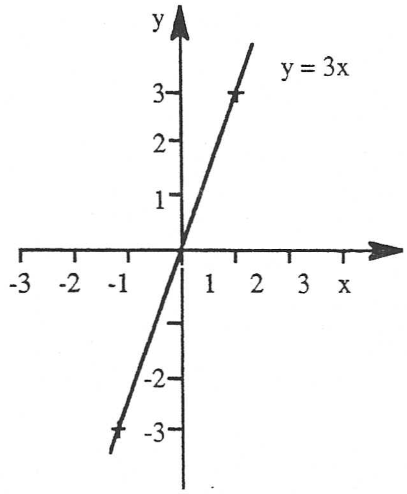
In this case x is the number of female engineers and y is the number of male ones.

Linear Equations

The word linear means referring to a line, so a linear equation represents a straight line drawn on a graph showing all solutions for y given values of x. The equation y=3x is represented below.

The line can be plotted by choosing three values for x and working out the corresponding values for y. The resulting **co-ordinates** are plotted on the graph.

x	-1	0	1
y	-3	0	3



The straight line graph y = 3x

When plotting graphs, the x and y axes can have different scales but if we wish to find from the graph, the points where it crosses the axes, they must always start from zero and progress positively and negatively.

Simplification

Sometimes equations are not given in their simplest form. We can rectify this by collecting together all the like terms.

eg $3a + 5b + 7a - 2b - 6c = 2a + b$

$10a + 3b - 6c = 2a + b$

This is still not in its simplest form because we have more than one term in a and b. By subtracting 2a from both sides we can combine the two terms in a, thus

$$(10a - 2a) + (3b - 6c) = b$$
$$8a + 3b - 6c = b$$

Similarly, subtracting b from both sides we arrive at the simplified result. These two operations could be performed at the same time, rather than separately as here.

eg. $8a + (3b - b) - 6c = 0$
$$8a + 2b - 6c = 0$$

Simplify the following equations.

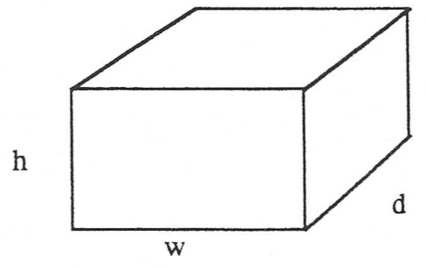
$2x + 3y - y = x$..
$x + y - z = x + 3y$...
$x^2 - x - 2x^2 + y = 0$...
$4x^3 - x^3 + 5x^2 = x^2$ ....

Substitution

Consider the box drawn below. Volume is represented by the formula

V = height x width x depth

Substituting the actual measurements for the box, we can find the volume.



If the box is a cube, then h = w = d so the volume is h x h x h = h^3. This is called the **index form** of h multiplied by itself three times. It saves time and is easier to read. Sometimes we write '.' instead of 'x', ie. h. h. h. , rather than h x h x h x.

If $x = 3$, $y = 4$ and $z = 2$ find the values of w below.

$w = x - 2y + 3z$..
$y - w = 5z + x$..
$z/y = w - z$..

Rewrite the following using index form.

s.s.t + s.t + t.t...
a.a.a.a - a.a.a.a.b...

[Solutions: -3, -9, 3.5; $s^2t + st$, $a^4(1 - b)$]

An equation is like a balance. Whatever you do to one side you must also do to the other side.

eg

$$\sqrt{x} = 3 \qquad 2y - 3 = 7$$
$$(\sqrt{x})^2 = (3)^2 \qquad 2y = 10$$
$$x = 9 \qquad y = 5$$

The Sraight Line Equation

In general a *linear equation* has the form $y = mx + c$ where m is the **gradient** of the straight line and c is the point where it crosses the y axis. m and c are called **constants** because they retain the same value whatever the value of x or y.

It is possible to find the equation of a straight line between two given points (x_1, y_1) and (x_2, y_2) using the formula below.

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

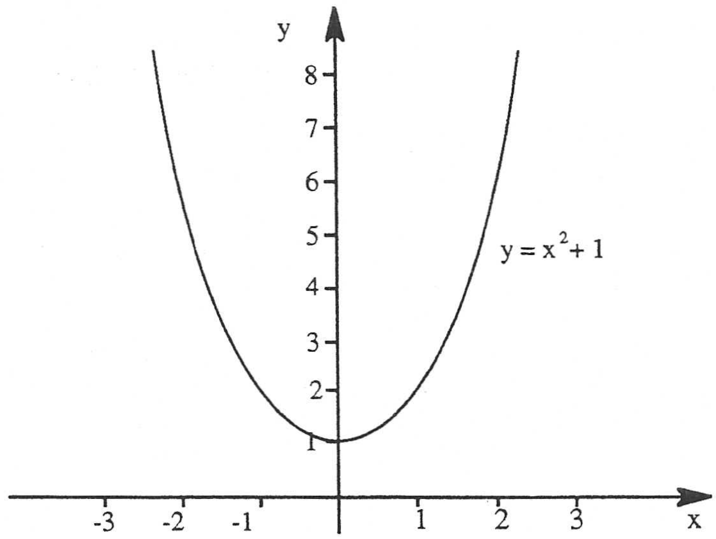
Quadratic Equations

This type of equation has the general form $y = ax^2 + bx + c$. They do not have straight line graphs, so when you plot them you need to take more than three values of x . Usually we take all the points in a given interval.

Consider the curve $y = x^2 + 1$

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
1	1	1	1	1	1	1	1
y	10	5	2	1	2	5	10

Example of a Quadratic curve



Notice that the curve is symetric about the y axis and it is always positive. It crosses the y axis at y = 1. These are points to look out for when plotting curves since they will help you to guess what the curve looks like before you start to plot it.

1.2 Algebraic Manipulation

Multiplication

Now that we have explained a little about quadratics we can begin to multiply equations. As with addition and subtraction, everything you do to one side of the equation you must also do to the other side.

Multiplying two linear equations together gives a quadratic. Multiplying three linear equations together gives a cubic and so on. These equations with higher powers are called **polynomials.**

When you multiply two linear equations together, each term in the first bracket is multiplied by each term in the other bracket.

eg $(x + 1)(x - 3) = x^2 - 3x + x - 3$

$= x^2 - 2x - 3$

This method of multiplication is also called **expansion** because the brackets are expanded into their most basic form.

> Remember that when multiplying
> 'minus' times 'minus' equals 'plus'
> 'minus' times 'plus' equals 'minus'

Expand the following brackets.

$(x + 1)(x - 1)$..

....

$(x - 3)(x + 3)$..

....

$(x^2 + 1)(x - 1)$..

....

$x(x + 1)(x - 1)$..

....

Hopefully you will have noticed that the brackets $(x + a)(x - a) = x^2 - a^2$.
This is called a *difference of two squares* and it is an important result which is valid whether a is a constant or a variable.

Factorisation

It is also possible to do the reverse of expansion. It is called factorisation and takes the equation from a basic form back to the bracketed **factors.**

Consider the equation below. The easiest way of factorising is to use the method given. Notice that the equation must be arranged so that it equals zero, this is called its *homogeneous* form.

eg $2x^2 - 6x - 20 = 0$

Given that the general quadratic equation is $ax^2 + bx + c = 0$, we need two constants m and n so that

$$\left.\begin{matrix} m \times n = ac = -40 \\ m + n = b = -6 \end{matrix}\right\} m = 4, n = -10$$

$$2x^2 + 4x - 10x - 20 = 0$$
$$2x(x + 2) - 10(x + 2) = 0$$
$$(2x - 10)(x + 2) = 0$$

This was easily done by taking common factors from each pair of terms, leaving the same bracket for each pair. The second factor is formed using the factors removed to form the first bracket.

Factorise the following equations.

$x^2 - 5x = -6$	$x^3 + x^2 - 6x = 0$
....	....
....	....
....	....
....	....
....	....

When a polynomial is in the form of a product of linear factors it can be solved to give a solution for x by equating each of the factors to zero so there is a solution for each factor. This is because when two numbers are multiplied together to give zero then either individual number must be zero, ie. a x b = 0 gives a = 0 or b = 0.

eg $(2x - 10) = 0$ $\qquad$ $x + 2 = 0$

$2x = 10$ $\qquad$ $x = -2$

$x = 5$

So for the example above, $x^3 + x^2 - 6x = 0$, the factors are x, (x + 2) and (x - 3). The solutions are therefore 0, -2 and 3 respectively. These solutions are sometimes called **roots** of the equation.

Find the roots of the following equations.

$x^2 + 5x + 6 = 0$	$2x^2 + 5x - 12 = 0$
....	....
....	....
....	....
....	....
....	....
	[Solutions: -2, -3; 3/2, -4]

Division

The numerical fraction 75/13 is called an **improper fraction,** its numerator (top number) is greater than its denominator (lower number). Long division can be used to turn it into a whole number plus a **proper fraction,** so that 75/13= 5 + 10/13. This process can be applied to algebraic fractions too. An algebraic function such as $\frac{3x^3 + 2x^2 + 1}{x + 1}$ looks better mathematically if it is reduced to a proper fraction ie one which is not top-heavy. This can be done in two ways, either we cancel out identical factors in the numerator and the denominator, much in the same way that 48/14 can be cancelled to 24/7, or we can use algebraic long division.

The numerator can be factorised and then cancelled to produce the same expression below.

eg $$\frac{3x^3 + 2x^2 + 1}{x + 1} = \frac{(3x^2 - x + 1)(x + 1)}{x + 1} = 3x^2 - x + 1$$

Sometimes when a polynomial is difficult to factorise, as this one was, or if it does not factorise using the method already explained, it is easier to use long division. This is just like ordinary long division using numbers except thet you must remember to include every term in the equation even if its coefficient is zero.

eg

$$
\begin{array}{r l}
 & 3x^2 - \ x + 1 \\
\cline{2-2}
x + 1 & \big) \, 3x^3 + 2x^2 + 0x + 1 \\
 & 3x^3 + 3x^2 \\
\cline{2-2}
 & \quad\quad - x^2 + 0x \\
 & \quad\quad - x^2 - x \\
\cline{2-2}
 & \quad\quad\quad\quad x + 1 \\
 & \quad\quad\quad\quad x + 1 \\
\cline{2-2}
 & \quad\quad\quad\quad\quad\ 0
\end{array}
$$

A zero remainder shows that the division is exact.

So we can see that $\dfrac{(3x^3 - 2x^2 + 1)}{x + 1} = 3x^2 - x + 1$

which can be checked by showing that $(3x^2 - x + 1)(x + 1) = 3x^3 + 2x^2 + 1$. In this case the remainder is zero but on dividng $3x^2 - x + 1$ by the factor $(x + 1)$ we do not get a zero remainder.

eg

$$
\begin{array}{r l}
 & 3x - \ 4 \\
\cline{2-2}
x + 1 & \big) \, 3x^2 - \ x + 1 \\
 & 3x^2 + 3x \\
\cline{2-2}
 & \quad\quad - 4x + 1 \\
 & \quad\quad - 4x - 4 \\
\cline{2-2}
 & \quad\quad\quad\quad\quad 5
\end{array}
$$

Now there is a remainder of 5, so we have

$$\frac{3x^2 - x + 1}{x + 1} = 3x - 4 + \frac{5}{x + 1}$$

or $\quad 3x^2 - x + 1 = (x + 1)(3x - 4) + 5$

and in general

$$\frac{\text{numerator}}{\text{denominator}} = \text{quotient} + \frac{\text{remainder}}{\text{denominator}}$$

or $\quad$ numerator = (quotient)(denominator) + remainder.

A useful tip when trying to factorise polynomials is to use the **remainder theorem** ie. for the above example, if $f(x) = 3x^2 - x + 1$ then $f(-1) = 3 - (-1) + 1 = 5$, so 5 is the remainder. If we substitute one of the roots of the equation, then the remainder will always be zero.

Completing the Square

Consider the equation $(x + 2)^2 = 0$. This is a complete square and it has two equal roots, -2 twice. All quadratics can be expressed as a square in the form $(x + k)^2 = c$ where c is a constant, k is not neccessarily a constant.

To complete the square of the equation $2x^2 + 5x - 12 = 0$ use the following method:

Step1: Make the x^2 coefficient equal to one.

$$2[x^2 + 5x/2 - 6] = 0$$

Step 2: The equation can now be written as

$$2\{ [x + 5/4]^2 - (5/4)^2 - 6 \} = 0$$

or $$[x + 5/4]^2 - (5/4)^2 - 6 = 0$$

This is always half of the number in front of x, and this is used to compensate for the extra $(5/4)^2$ introduced by the $[x + 5/4]^2$ bracket that wasn't there originally!

Step 3: Rearrange the equation so

$$[x + 5/4]^2 = 25/16 + 6$$
$$[x + 5/4]^2 = 121/16$$

The equation can now be solved .

$[x + 5/4] = \pm 11/4$ c^2 can be c x c or (-c)(-c)

$x = 6/4$ or $x = -16/4$

$x = 3/2$ $x = -4$

Complete the square and solve the following equations.

$x^2 + 4x + 1 = 0$ $ax^2 + bx + c = 0$

....

....

....

....

....

[Soln: $(x - 2)^2 = 3$, $x = 2 \pm \sqrt{3}$; $a(x+b/2a)^2 = b^2/4a$, $x = [-b \pm \sqrt{(b^2 - 4ac)}]/2a$]

1.3 The Nature of Equations

The solution to the second problem above is very important since it is the solution of the general quadratic equation. If a quadratic cannot be factorised immediately, the formula below may be used. It is derived from the problem above.

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

By considering $\Delta = b^2 - 4ac$ it is possible to determine the nature of the roots of an equation.

If $\Delta < 0$	no real roots (because the square root of a negative number does not exist).
If $\Delta = 0$	two equal roots
If $\Delta > 0$	two distinct roots.

Suppose the roots of a quadratic were α and β, what is the equation?

Working backwards we have

$$(x - \alpha)(x - \beta) = 0$$
$$x^2 - \beta x - \alpha x + \alpha\beta = 0$$
$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

So in general the equation whose roots are α and β is

x^2 - x(sum of roots) + (product of roots) = 0
note the minus sign!

If an equation has roots -1 and 2 what is the equation?

$$\alpha + \beta = -1 + 2 = 1$$
$$\alpha\beta = (-1)^2 = -2$$

The equation is $x^2 - x - 2 = 0$

If the equation $x^2 + 3x - 5 = 0$ has roots α and β what is the equation whose roots are α^2 and β^2 ? We know that $\alpha + \beta = -3$ and $\alpha\beta = -5$, but we do not know about α^2 or β^2.

Now $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

$\alpha^2 + \beta^2 =$ sum of roots $= (\alpha + \beta)^2 - 2\alpha\beta$

and $(\alpha\beta)^2 =$ product of roots $= \alpha^2\beta^2$

So $(\alpha^2 + \beta^{2)} = (-3)^2 - 2(-5) = 19$

$\alpha^2\beta^2 = (-5)^2 = 25$

The equation is therefore $x^2 - 19x + 20 = 0$

Summary

Mathematical models may be used to solve everyday problems. These problems are often in the form of mathematical equations because they are such an ideal way of finding an unknown quantity. The way to model such a problem is

- State the problem.
- Formulate a mathematical equation for it.
- Solve the problem.
- Translate the solution back into the context of the problem.

We have seen how it is possible to manipulate equations as if they were actual numbers. They can be added, subtracted and multiplied by a factor just like any other number as long as the equation remains balanced.

As well as algebraic manipulations, we can carry out multiplication and division of two equations. This is essentially expansion and factorisation to enable us to express the equation in the simplest or most convenient form possible.

We have also investigated the essential nature of quadratic equations and how they are related, by looking at the roots. Further investigations into the general equation are contained in the activities which follow.

Activities

1. Formulate an equation for the following statement. 'Andy is three years older than his brother Nick who is half the age of their father'.

2. Plot the points A(2, 3), B(4, 0) and C(-4, -1) on a graph and find the equations of the three lines which form the sides of the triangle ABC.

3. Solve the following linear equations.

   (a) $2x + 3x - 7x = 6$ (b) $5v + 7v = 0$

   (c) $12y = -144$ (d) $y + y - 3y = 16$

4. Expand the following equations.

   (a) $y = (x + 3)(x - 1)$ (b) $y = (x - 6)(2 - x)$

   (c) $y = x(x + 3)(x - 7)$ (d) $y = (4x^2 + 1)(2 - x)(x + 5)$

5. Plot the graphs of the following equations for values of x between -4 and 4.

   (a) $y = x^2 - 3x + 1$ (b) $y = x - 1$

   (c) $y = x^3$ (d) $y = x^4$

   (e) $y = -x^2$ (f) $y = x^3 + 2x^2 - x - 1$

6. What is the point of intersection of the two curves below? Show this by plotting them on the same set of axes.

   $y = 2x^2 - 7x - 17$
   $y = 2x^2 + 9x - 13$

7. Factorise and solve the following equations completely.

(a) $x^2 - 5x + 4 = 0$	(b) $6x^2 - x - 1 = 0$
(c) $4x^3 - 2x^2 - 2x = 0$	(d) $x^4 - 1 = 0$
(e) $2x^3 + x^2 - 2x - 1 = 0$	(f) $x^2 + x - 1 = 0$
(g) $2x^2 + 3x - 1 = 0$	(h) $x^2 + 3x - 1 = 0$

8. Find the remainder on dividing

(a) $x^2 - 5x + 4$ by $(x - 2)$

(b) $y = \dfrac{x^3 + x^2 + x + 1}{x + 1}$

(c) $y = x^4 + x^2 + 1$ by $y = x + 2$

9. For what values of k does the equation below have real roots?

$$(x - 5)(x + 1) = k(x - 7)$$

10. What is the equation whose roots are α, β and γ?
What is the equation whose roots are α, β, γ and δ?
Is there a pattern emerging?

11. If the equation $x^2 + 3x - 5 = 0$ has roots α and β, what is the equation whose roots are $1/\alpha$ and $1/\beta$?

[Solutions: **1** $A = 2N - C/2 + 3$.

2 AB , $y = 6 - 3x/2$; AC, $y = 2x/3 + 5/3$; BC, $y = x/8 - 1/2$.

3 (a) -3; (b) 0; (c) -12; (d) -16.

4 (a) $y = x^2 + 2x - 3$; (b) $y = -x^2 + 8x - 12$; (c) $y = x^3 - 4x^2 - 21x$;
(d) $y = -4x^4 - 12x^3 + 39x^2 - 3x + 10$.

6 (-1/4, -121/8).

7 (a) $x = 1, 4$; (b) $x = 1/2, -1/3$; (c) $x = 0, -1/2, 1$; (d) $x = 1$ twice, -1 twice;
(e) $x = 1, -1, -1/2$; (f) $x = -1/2 \pm \sqrt{5}/2$; (g) $x = -3/4 \pm \sqrt{17}/4$; (h) $-3/2 \pm \sqrt{5}/2$.

8 (a) -2; (b) 0; (c) 21.

9 $k \leq 2$ and $k \geq 18$.

10 $x^3 - x^2(\alpha + \beta + \gamma) + x(\alpha\beta + \alpha\gamma + \beta\gamma) - \alpha\beta\gamma = 0$,
$x^4 - x^3(\alpha + \beta + \gamma + \delta) + x^2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) - x(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) + \alpha\beta\gamma\delta = 0$.

11 $5x^2 - 3x - 25 = 0$.]