2. ALGEBRAIC METHODS

Objectives

To be able to solve two equations simultaneously, whether linear or quadratic using algebraic methods and graphical methods.

To use the method of partial fractions and understand that it is the reverse of adding algebraic fractions.

To understand that solutions may be in the form of a group of numbers and to be able to represent the group algebraically using inequalities.

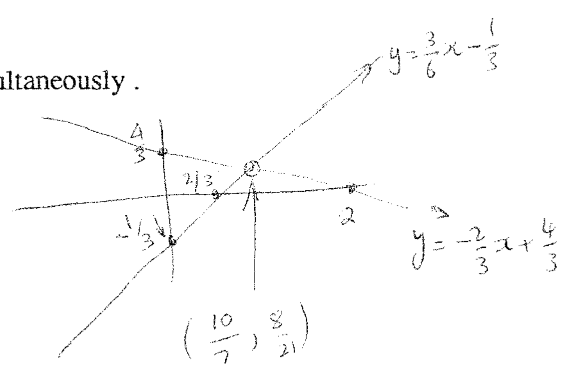
2.1 Simultaneous Equations

Linear Equations

Given two equations it is possible to solve them simultaneously .

eg $2x + 3y = 4$ (1) → $y = -\frac{2}{3}x + \frac{4}{3}$

$3x - 6y = 2$ (2) → $y = +\frac{3}{6}x - \frac{2}{6}$



Multiply (1) by two so we have

$4x + 6y = 8$ (1)

$3x - 6y = 2$ (2)

Now add the two equations.

$7x + 0y = 10$

$x = 10/7$

The first solution is $x = 10/7$. Substitute this back into either (1) or (2) to find the second solution.

ie $4(10/7) + 6y = 8$

$6y = 16/7$

$y = 8/21$

So the complete solution is (10/7, 8/21). This means that both of these equations hold for these values.

If one of the equations is a quadratic, it is easier to solve them simultaneously if we equate them to each other.

eg $y = x^2$ (1)

$y = x + 2$ (2)

since y is equal to both x^2 and $x + 2$, then $x^2 = x + 2$

$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = 2, \text{ or } x = -1$$

By solving the quadratic we obtain two solutions for x. The corresponding solutions for y are found as before.

eg $y = (2)^2 = 4$ and $y = (-1)^2 = 1$

So (2, 4) and (-1, 1) are the two solutions required.

Equations of Higher Power

If both the equations are quadratic, the best method of solution is to equate them to each other.

Solve the following pair of simultaneous equations.

$y = x^2 - x - 2$

$y = 2x^2 - 4x - 6$

....

....

....

....

....

....

....

....

....

[Solution: (-1, 0), (4, 10)]

Consider the pair of simultaneous equations below. They can be solved using the same method as was used for the pair of linear equations.

$$3x^2 - 4y^2 = 11 \qquad (1)$$
$$x^2 + y^2 = 13 \qquad (2)$$

(2) x 3

$$3x^2 - 4y^2 = 11 \qquad (1)$$
$$3x^2 + 3y^2 = 39 \qquad (2)$$

(2) - (1)

$$7y^2 = 28$$
$$y^2 = 4$$
$$y = \pm 2$$

From (2)

$$x^2 + 4 = 13$$
$$x^2 = 9$$
$$x = \pm 3$$

The solutions are therefore (3, 2), (-3, 2), (3, -2) and (-3, -2).

Now consider the following pair of simultaneous equations. How do we solve them without getting lost in arithmetic?

$$x^2 + y^2 = 13 \qquad (1)$$
$$xy = 6 \qquad (2)$$

Now, $y = 6/x$

$$\text{so } x^2 + (6/x)^2 = 13$$
$$x^2 + 36/x^2 = 13$$
$$x^4 + 36 = 13x^2$$
$$x^4 - 13x^2 + 36 = 0$$

This can be seen to be a quadratic in x^2 if x^2 is written as A, say. Then $A^2 - 13A + 36 = 0$.

What are the **four** solutions to this equation?

....

....

....

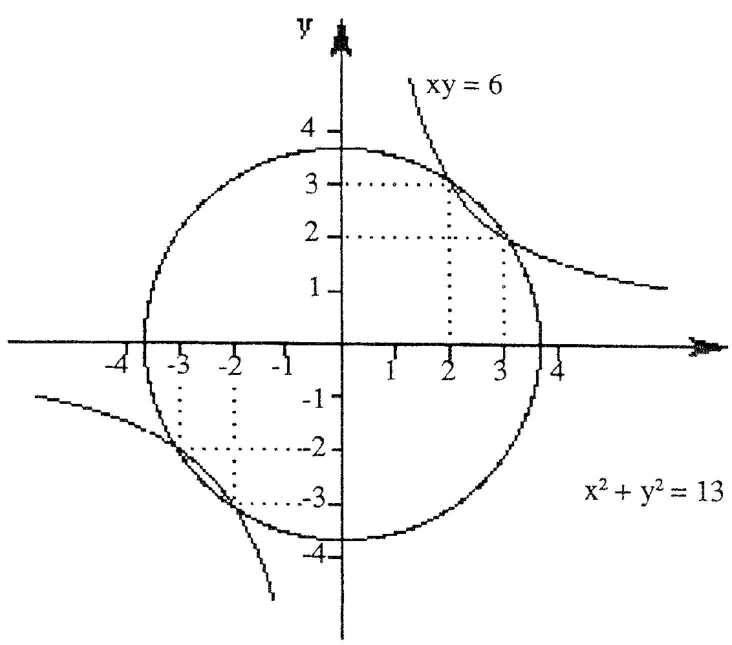
....

....

[Solutions: (3, 2), (2, 3), (-2, -3) and (-3, -2)]

Graphical interpretation is another method of solving a pair of simultaneous equations, however the graph must be very accurate to do this. The solutions are at the points where the lines or curves of the equations intersect. Below is the graph of the previous example.

An example of the graphical solution of two equations



2.2 Partial Fractions

Revision of Fractions

Fractions may be added if they have a common denominator.

ie $4/5 + 2/3 = \frac{3 \times 4 + 5 \times 2}{15}$

$= \frac{12 + 10}{15}$

$= 17/15$

We can do the same with algebraic fractions.

eg $\frac{1}{x+1} + \frac{2}{x} = \frac{x + 2(x + 1)}{x(x + 1)}$

$= \frac{3x + 2}{x(x + 1)}$ this is called a **compound fraction**

But how do we separate the compound fraction back into two **partial** fractions, each with a linear term in x as the denominator? We need to be able to reverse the process we used when adding partial fractions.

No Repeated factor in the Denominator

Express in partial fractions the function $f(x) = \frac{3x - 2}{(x + 1)(2x - 5)}$

We write $\frac{3x - 2}{(x + 1)(2x - 5)} \equiv \frac{A}{(x + 1)} + \frac{B}{(2x - 5)}$ in partial fractions

Note the ≡ sign is always used to indicate that this is an **identity** - true for all values of x.

Multiplying both sides by (x + 1)(2x - 5) gives us

$$3x - 2 \equiv A(2x - 5) + B(x + 1)$$

Now we want to make the coefficient of A equal to zero so that we can find out what B equals. Then we do the same for B to find out A.

ie $2x - 5 = 0$ so $x = 5/2$ $15/2 - 2 = 7B/2$

$B = 11/7$

Now let $x = -1$ $-3 - 2 = -7A$

$A = 5/7$

Substituting back into the partial fractions we have,

$$f(x) = \frac{3x - 2}{(x + 1)(2x - 5)} \equiv \frac{5}{7(x + 1)} + \frac{11}{7(2x - 5)}$$

eg. Express in partial fractions $f(x) = \dfrac{2x^2 - x + 3}{(x + 1)(x - 2)(x + 3)}$

$$f(x) \equiv \frac{A}{(x + 1)} + \frac{B}{(x - 2)} + \frac{C}{(x + 3)}$$

$$2x^2 - x + 3 \equiv A(x - 2)(x + 3) + B(x + 1)(x + 3) + C(x + 1)(x - 2)$$

We want to eliminate terms involving A and B to find C.

let $x = -3$ $24 = 10C$

$C = 12/5$

Similarly,

let $x = 2$ $9 = 15B$ let $x = -1$ $6 = -6A$

$B = 3/5$ $A = -1$

$$f(x) = \frac{2x^2 - x + 3}{(x + 1)(x - 2)(x + 3)} \equiv \frac{-1}{(x + 1)} + \frac{3}{5(x - 2)} + \frac{12}{5(x + 3)}$$

Non-linear Factors

Express in partial fractions $f(x) = \dfrac{x^2 + 2}{(x^2 + 1)(x - 1)} \equiv \dfrac{Ax + B}{(x^2 + 1)} + \dfrac{C}{(x - 1)}$

The denominator in the first partial fraction cannot be split into linear factors, so we introduce a numerator of the form Ax + B.

$$x^2 + 2 \equiv (Ax + B)(x - 1) + C(x^2 + 1)$$

let x = 1 $\quad 3 = 3C$ $\quad$ let x = 0 $\quad 2 = -B + C$

$C = 1$ $\qquad B = -1$

let x = 2 $\quad 6 = 2A + B + 5C$

$6 = 2A - 1 + 5$

$A = 3/4$

$$f(x) = \frac{x^2 + 2}{(x^2 + 1)(x - 1)} \equiv \frac{(3x/4) - 1}{x^2 + 1} + \frac{1}{x - 1}$$

$$\equiv \frac{3x - 4}{4(x^2 + 1)} + \frac{1}{x - 1}$$

Repeated Factors

Express the following in partial fractions, $f(x) = \dfrac{x - 1}{(2x - 3)^2}$

Notice that $\dfrac{f(x)}{(x - a)^n} \equiv \dfrac{A_1}{x - a} + \dfrac{A_2}{(x - a)^2} + \ldots + \dfrac{A_n}{(x - a)^n}$ so that the partial fractions are as below.

$$f(x) = \frac{x - 1}{(2x - 3)^2} \equiv \frac{A}{2x - 3} + \frac{B}{(2x - 3)^2}$$

$$x - 1 \equiv A(2x - 3) + B$$

let x = -3/2 $\quad B = -5/2$ $\quad$ let x = 0 $\quad -1 = -3A + B$

$3A = -3/2$

$A = -1/2$

$$f(x) = \frac{x - 1}{(2x - 3)^2} \equiv \frac{-1}{2(2x - 3)} - \frac{5}{2(2x - 3)^2}$$

If a fraction is top-heavy then it is improper and must be divided out before it can be expressed in partial fractions.

Express the following top-heavy fraction as the sum of a polynomial and a proper fraction.

$$f(x) = \frac{x^3 + 1}{x^2 - x}$$

....

....

....

....

....

[Solution: $f(x) = 1 + x + (x + 1) / (x^2 - x)$]

The example above can still not be written in partial fractions. The denominator has yet to be factorised.

ie $\quad x^2 - x = x(x - 1)$

$$f(x) = 1 + x + \frac{x + 1}{x(x - 1)}$$

$$\frac{x + 1}{x(x - 1)} \equiv \frac{A}{x} + \frac{B}{x - 1}$$

$$x + 1 \equiv A(x - 1) + Bx$$

let $x = 1$ $\quad 2 = 2B$, $\quad B = 1$ $\qquad$ let $x = 0$ $\quad 1 = -A$, $\quad A = -1$

$$f(x) = 1 + x + \frac{x + 1}{x(x - 1)} \equiv 1 + x - \frac{1}{x} + \frac{1}{(x - 1)}$$

2.3 Inequalities

For all real numbers we can say one of the following statements.

$x = 0$	x equals zero
$x < 0$	x is negative
$x > 0$	x is positive

If we use the equals sign we have an equation. If we use the greater than or less than signs we have an **inequality.**

eg $a - 3 = 10$ is an equation
$a + 2 > 7$ is an inequality.

The signs can be developed to include $\leq$, $\geq$, $\neq$, $\nless$, $\ngtr$, $\nleq$, $\ngeq$ so that the following mean the same thing.

$x \nless 3$ $\quad$ $x \geq 3$

$>$ and $<$ mean *strictly* greater than and *strictly* less than.

ie If x is an integer, $x > 5$ means integers from 6 to infinity, not including 5
$x \geq 5$ means integers from 5 to infinity, including 5.

If x satisfies two inequalities we can combine them into a single statement.

ie $x < 5$ and $x > 3$ can be written $3 < x < 5$.
Note that this could have been , but is usually not, written as $5>x>3$ since these ' multiple' inequalities are usually written 'smaller value $<$ x $<$ larger value'.

It is useful to use a number line when dealing with inequalities, especially when the algebra becomes difficult.

Rules of Inequalities

- When the same number is added to both sides of an inequality it is unchanged.
- Both sides may be multiplied or divided by the same *positive* number.
- The inequality sign *reverses* when both sides are multiplied or divided by the same **negative** number.

eg	$8 > 6$	divide both sides by -2
	$-4 > -3$	this is clearly incorrect. (Draw a number line to check this)
	$-4 < -3$	this is correct.

Solve these simple inequalities.

$x + 7 > 22$ $\qquad$ $2 - 5x \leq 7$ $\qquad$ $1 \geq -5x + 2$

....

....

....

[Solutions: $x > 15$, $x \geq -1$, $x \geq 1/5$]

The Modulus Sign

The absolute value or modulus of x is given by

$|x| = \sqrt{(x^2)}$ so that $|x| = x$ $\qquad$ if $x \geq 0$

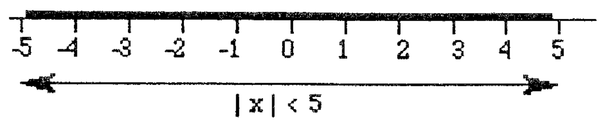
$|x| = -x$ $\qquad$ if $x \leq 0$

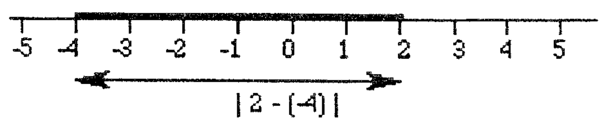
ie $\quad |2| = 2 \qquad |-2| = 2$

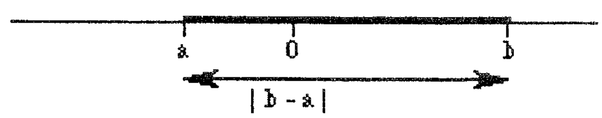
On the number line $|x|$ is the distance of x from zero and $|x - y| = |y - x|$ is the distance between points x and y. This means that there are really two inequalities where a modulus is used.

ie $\quad |x| < 5$ $\quad$ is really $\quad -5 < x < 5$

Some examples are shown below.







Solve the inequality $|2x - 3| < 7$

....

....

....

....

[Solution: $-2 < x < 5$]

Uses of Inequalities

Solve the inequality $\frac{(2x - 5)}{x} \leq 7$

$$\frac{(2x - 5)}{x} \leq 7$$

$$0 \leq 7 - \frac{(2x - 5)}{x}$$

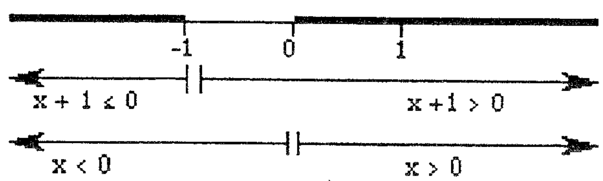
$$0 \leq \frac{7x - 2x + 5}{x}$$

$$0 \leq 5 \left[\frac{x + 1}{x} \right]$$

The bracket is positive when both the numerator and the denominator are the same sign. By looking at the number line below you will see when this can happen.

$x + 1 \leq 0$ and $x < 0$ or $x + 1 \geq 0$ and $x > 0$

$x \leq -1$ $x \geq -1$



In order to satisfy all these conditions we must choose $-1 \geq x > 0$. In general, when there is a choice of boundaries, such as here, we choose the most extreme values so that all conditions are satisfied.

Alternatively consider this method. The terms in the bracket change sign either side of $x = -1$ (numerator) and $x = 0$ (denominator). mark these values on a number line and choose any value, say $x = 2$ (in the interval $x > 0$). In this case the inequality is true, so it must be true in the whole of this interval. It can easily be shown that the inequality is true in alternate intervals, so it must also be true when $x \leq -1$. So $x > 0$ or $x \leq -1$.

BEWARE: The inequality sign adopted in the answer is the same as that in the original problem, but watch out for division by zero. So here x cannot take the value of zero the answer is therefore $x > 0$ and not $x \geq 0$.

Solve the inequality $\frac{2x + 1}{x} < 4$

$$\frac{2x + 1}{x} - 4 < 0$$

$$\frac{2x + 1 - 4x}{x} < 0$$

$$\frac{1 - 2x}{x} < 0$$

This is satisfied when the left hand side is negative, ie when the numerator and the denominator have opposite signs.

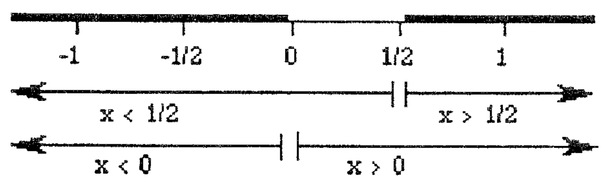
$1 - 2x < 0$ and $x > 0$

$1/2 < x$

or

$1 - 2x > 0$ and $x < 0$

$1/2 > 0$



The inequality is satisfied for $0 > x > 1/2$.

Quadratic Inequalities

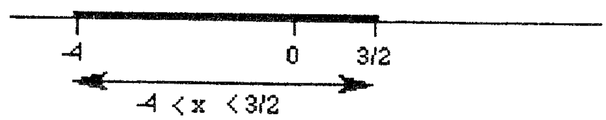
Quadratic inequalities may be a little more difficult but they are based on the same principle.

eg Solve the inequality $2x^2 + 5x - 12 < 0$.

By factorising, we arrive at something which is easier to analyse.

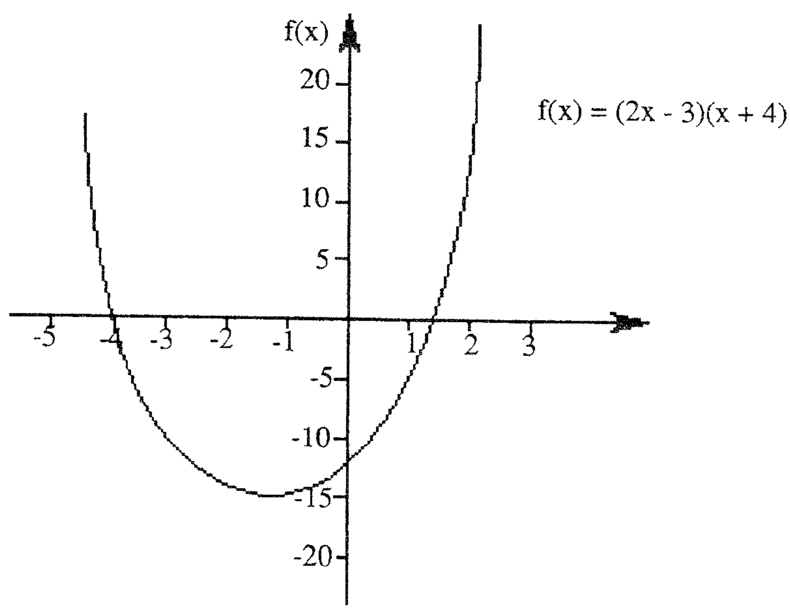
$(2x - 3)(x + 4) < 0$

By inserting values of x for each interval we find the solution range.



Clearly the only solution for $2x^2 + 5x - 12 < 0$ is when $-4 < x < 3/2$ since it is the only range when f(x) is negative. This can be seen quite easily from the graph. The curve lies below the x axis between the values of $x = -4$ and $x = 3/2$.

Graphical solution to an inequality



If the inequality does not factorise, it is best to use the method of completing the square to find out how it behaves.

eg $\quad f(x) = x^2 + x - 1 \geq 0$

suppose $\quad (x + 1/2)^2 - 5/4 = 0$

$$(x + 1/2)^2 = 5/4$$

$$x + 1/2 = \pm\sqrt{(5/4)}$$

$$x = -1/2 \pm \sqrt{5}/2$$

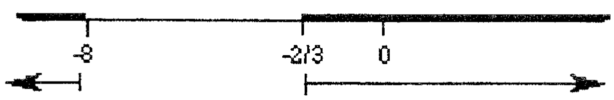
Now for $x \geq -1/2 + \sqrt{5}/2$, $\quad f(x) > 0$

$x \leq -1/2 - \sqrt{5}/2$, $\quad f(x) > 0$

Sometimes there may be a modulus included in the inequality. The factor contained within the modulus is squared and then the inequality is treated as any other.

eg
$$|x - 3| < |2x + 5|$$
$$(x - 3)^2 < (2x + 5)^2$$
$$x^2 - 6x + 9 < 4x^2 + 20x + 25$$
$$3x^2 + 26x + 16 > 0$$
$$(3x + 2)(x + 8) > 0$$

Substituting values of x from each of the intervals, we obtain the region below.



Clearly $|x-3| < |2x + 5|$ for $x < -8$ and $x > -2/3$.

Summary

This unit has introduced a number of algebraic methods which are used in more advanced topics.

Simultaneous equations may be regarded as being curves on the same graph, with solutions at the point where they meet. The methods introduced here can be used for any pair of equations. The same sort of methods can be used to solve more than two equations simultaneously. By eliminating variables and equations successively you can solve a *system* of equations.

Partial fractions are used to split a single algebraic compound fraction so that we have two or more separate fractions, each with a *single factor* in the denominator. This method is used later when differentiating, integrating and using the Laplace transform because it simplifies the calculations in the long run.

Inequalities are used to indicate a solution region and the method is a lot more efficient than using a graph to see the required region. However it is still useful to draw the graph or the number line, for complex inequalities because it can be easy to overlook solutions or choose the wrong region. You will find that topics you may study later on will take solution of inequalities for granted.

Acivities

1. Solve the following simultaneous equations.

(a) $3x - 5y = 8$
$2x + y = 1$

(b) $4x + 7x + y = 5$
$2y - 3y + x = 7$

(c) $y = x + 1$
$y = x^2 + 3x - 2$

(d) $y - x^2 = 3x + 1$
$y = 2x + 3$

(e) $6x^2 + 2y^2 - 7 = 0$
$3x^2 - 2y^2 - 2 = 0$

(f) $x/5 + y/2 = 4$
$(x - 1)/3 = (y + 2)/2$

2. Solve the following simultaneous equations graphically.

(a) $4x + 3y = 29$
$y = x + 5$

(b) $2x + 3 = 7 - y$
$5x - 2y = 4x - 3$

(c) $6x - 2y = 16$
$3x - y = 4$

(d) $6x + 3y = 9$
$4x + 2y = 6$

3. Express the following as the sum of a proper polynomial and partial fractions.

(a) $\dfrac{x^3 + 1}{x - 1}$ (b) $\dfrac{x^3 - 5x^2 + 3x}{(x + 1)(x - 2)}$ (c) $\dfrac{x^3 + x^2}{(x^2 + x + 2)(x - 3)}$

4. Express the following in partial fractions.

(a) $\dfrac{x}{(x + 2)(x + 3)}$ (b) $\dfrac{x^5}{x^4 - x^2}$ (c) $\dfrac{x - 1}{x(x + 1)^2}$

(d) $\dfrac{3}{(2x + 1)(x + 1)}$ (e) $\dfrac{x^3 + 6}{(x - 1)(x + 2)(x - 3)}$

5.Solve the following inequalities.

(a) $\frac{5x - 4}{x} \leq 5$ (b) $\frac{2x + 3}{x} \geq 2$

(c) $\frac{3x - 4}{x + 2} < 1$ (d) $\frac{5 - 2x}{4 - 3x} < 3$

6.Find the values of x which satisfy the following.

(a) $|x - 2| > 3$ (b) $|x - 2| > |x + 2|$ (c) $|x - 2| > 5|x - 3|$

7.Find the values of x which satisfy $x^2 - 4x + 3 > 0$.

[Solutions:

1 (a) (1, -1); (b) (1, -6); (c) (1, 2), (-3, -2); (d) (1, 5), (-2, -1); (e) (1, 1/√2), (-1, 1/√2); (f) (10, 4).

2 (a) (2, 7); (b) (1, 2); (c) parallel; (d) coincide.

3 (a) $x^2 + x + 1 + 2/(x + 1)$; (b) $x - 4 + 1 + 1/3(x + 1) + 2/3(x - 2)$; (c) $(3x - 2)/ 7(x^2 + x + 2) + 18/7(x - 3)$.

4 (a) $-2/(x + 2) + 3/(x + 3)$; (b) $x + 1/2(x + 1) + 1/2(x - 1)$; (c) $-1/x + 1/(x + 1) + 2/(x + 1)2$; (d) $6/(2x + 1) - 3/(x + 1)$; (e) $1 - 7/6(x - 1) - 2/15(x + 2) + 33/10(x - 3)$.

5 (a) $x > 0$; (b) $x > 0$; (c) $-2 < x < 3$; (d) $4/3 < x < 1$.

6 (a) $x < 1, x > 3$; (b) $x < 0$; (c) $17/6 < x < 13/4$.

7 $1 > x > 3$.]