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Principles of Wireless Sensor Networks

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Lecture 12

Wireless Sensor Network Control Systems 1

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Course content

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  - Lec 3: Physical Layer
  - Lec 4: Medium Access Control Layer
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Previous lecture

Application
Presentation
Session
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Routing
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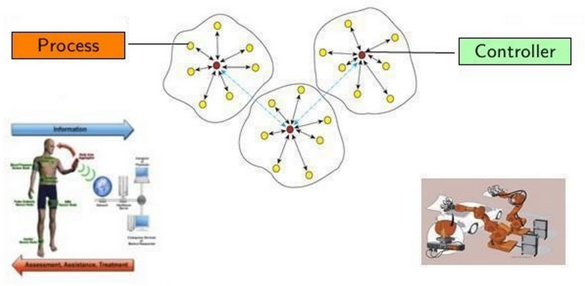
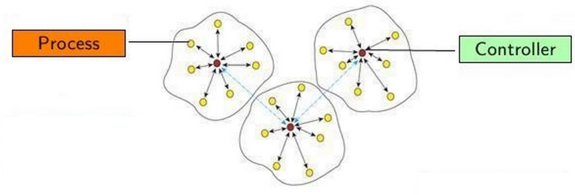
Phy



How to synchronize nodes?

Today's learning goals



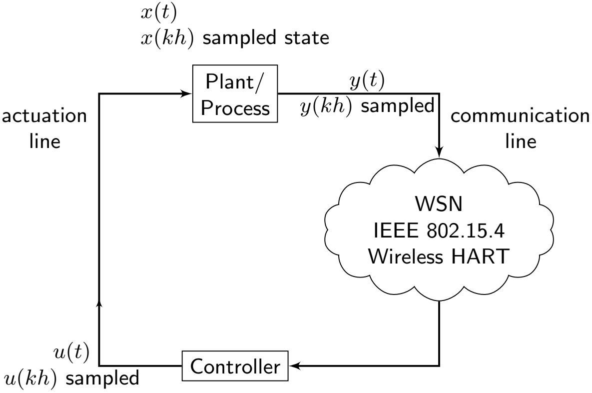
- How the state dynamics over time are mathematically modeled?
- How such state dynamics can be controlled by closing the loop process→controller→process?
- How to discretize the continuous time model of the dynamics?
- What is the concept of state stability of closed loop control systems?
- How the stability of closed loop control systems is affected by constant network delays?

Outline

- Wireless Sensor Network Control Systems (WSNCS)
- State space description of a control system
- Stability and asymptotic stability of a control system
- Stability of a control system in the presence of constant network delays

Wireless Sensor Network Control Systems (WSNCS)

Closed-loop system



Wireless Sensor Network Control Systems

k: discrete time

h: sampling interval

- $u(kh)$: control decision
- $x(kh)$: state of the process/plant
- $y(kh)$: output of the state (measured by sensors)

- The **GOAL** of the controller is to bring the state $x(kh)$ in a desired region by taking measurements $y(kh)$ and a control decision $u(kh)$
- Delay and packet loss probability affect the way the measurements $y(t)$ are received in the controller

This lecture gives the basic control theory background for WSNCS

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- Wireless Sensor Network Control Systems (WSNCS)
- State space description of a control system
  - Linear model
    - Continuous time description
    - Discretization of state space model
  - Non-linear model
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Continuous time description

Let $x(t)$ be the state (temperature, position, pollution...). We assume that the physical process is described by the time-invariant state space model

Linear model

$$\begin{aligned} \frac{dx(t)}{dt} \triangleq \dot{x}(t) &= Ax(t) + Bu(t) \qquad \text{state model} \\ y(t) &= Cx(t) + Du(t) \qquad \text{measurement model} \end{aligned} \tag{1}$$

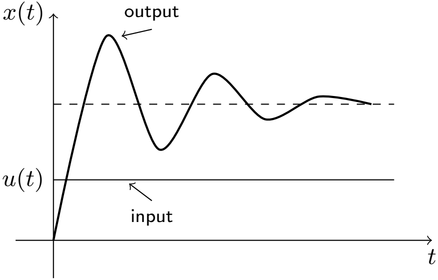
where A, B, C, D are assumed to be known matrices

Example: "The step response"

Suppose $x(0) = 0$ and $x(t) \in \mathbb{R}$

The step response is defined as the solution of (7) when we apply as input

$$u(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$



The state may evolve to a stabilized condition after possible oscillations

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Discretization of state space model

Assuming that $x\left(kh\right)$ is known, the solution of the simple differential equation (1) is

$$x\left(t\right) = e^{A(t-kh)} \cdot x\left(kh\right) + \int\limits_{kh}^{t} e^{A(t-\tau)} Bu\left(t\right) d\tau \qquad t > kh \tag{2}$$

Note that $u\left(t\right)$ can be properly chosen so that $x\left(t\right)$ falls in a desired region

Discretization of state space model

Assume $u(t)$ constant in the interval $kh \leq t \leq kh + h$

Then, (10) becomes

$$x(t) = e^{A(t-kh)} \cdot x(kh) + \int_{kh}^{t} e^{A(t-\tau)} d\tau B u(t) =$$

$$= e^{A(t-kh)} \cdot x(kh) + \int_{0}^{t-kh} e^{A\tau} d\tau B u(kh) = \phi_t x(kh) + \Gamma_t u(kh)$$

Let $t = kh + h$

$$x(kh+h) = \phi x(kh) + \Gamma u(kh) \qquad (3)$$

where $\phi = e^{Ah}$ and $\Gamma = \int_{0}^{h} e^{A\tau} d\tau B$

Discretization of state space model

There are many ways to compute e^{Ah}, for example

$\phi = e^{Ah} = I + Ah + \frac{A^2h^2}{2} + \dots$

Recursively from (3),

$$x\left(kh + 2h\right) = \phi x\left(kh + h\right) + \Gamma u\left(kh + h\right)$$

Therefore, the solution of (3), given $x\left(0\right)$ and $u\left(kh\right) \quad \forall k$, is

$$x\left(kh\right) = \phi^k x\left(0\right) + \sum_{j=0}^{k-1} \phi^{k-1-j} \Gamma u\left(jh\right)$$

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Non-linear model of the state

Observation

Control decision is chosen as a function of the state

$$u(t) = f(x(t))$$

Therefore, consider a state that evolves according to a non-linear law

$$\dot{x}(t) = a(x(t))$$
$$y(t) = c(x(t))$$

where a and c are be non-linear functions in general

What is the solution of that system?

Non-linear model of the state

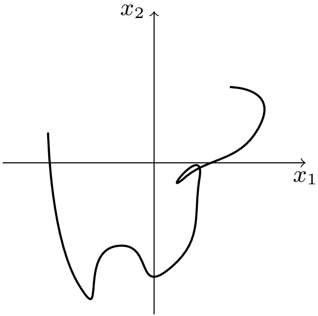
Non-linear differential equation

$$x\left(t+kh\right)=x\left(t\right)+\int_{t}^{t+kh}a\left(x\left(\tau\right)\right)d\tau$$

- In general, the integral may be difficult to solve

Example: a non linear motion

Movement of an object (e.g., mosquito motion)



$$\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{x}_2 &= v_2 \\
\dot{v}_1 &= -\omega v_2 \\
\dot{v}_2 &= \omega v_1 \\
\dot{\omega} &= 0
\end{aligned}$$

- The parameter ω determines the movement
- $\dot{v}_1 = -\omega v_2 \Rightarrow$ Non-linear

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Stability

Let us consider the discrete-time differential equation

$$x\left(kh+h\right)=g\left(x\left(kh\right)\right) \tag{4}$$

where g can be linear or non-linear

Definition

A specific solution of (4), $x^*\left(kh\right)$, is called stable,
if $\forall\varepsilon>0 \quad \exists\delta\left(\varepsilon\right):$ $\forall$ other solution $x\left(kh\right)$

$$\left\|x\left(0\right)-x^*\left(0\right)\right\|\leq\delta\Rightarrow\left\|x\left(kh\right)-x^*\left(kh\right)\right\|\leq\varepsilon \quad \forall k$$

Asymptotic stability

We consider the same equation as on the previous slide:

$$x(kh + h) = g(x(kh)) \tag{5}$$

where g can be linear or non-linear.

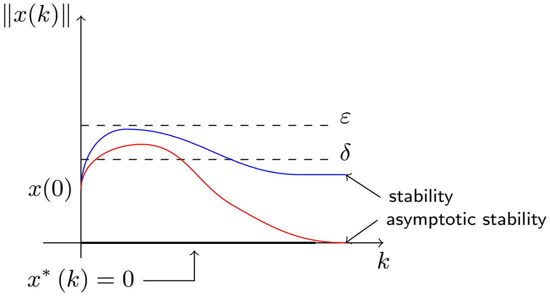
Definition

A specific solution $x^*(k)$ of (5) is called **asymptotically stable** if it is stable and if there is a $\delta > 0$ such that for every other solution $x(k)$ it holds that:

$$\|x(0) - x^*(0)\| \leq \delta \Rightarrow \|x(k) - x^*(k)\| \to 0 \quad as\ \ k \to \infty$$

Example 1

Assume that $x^*(kh) = 0$ is a solution of (5). The figure shows the typical behaviour of other solutions in case $x^*(kh)$ is stable or asymptotically stable.



The linear case

Consider a linear case, that is

$$x\,(kh+h) = A \cdot x\,(kh) \tag{6}$$

where A known matrix $\in \mathbb{R}^{n\times n}$

Definition

A linear difference equation of the form (6) is (asymptotically) stable if the constant solution $x^*(k) = 0$ is (asymptotically) stable.

How do we choose matrix A in order to have

1. stability?
2. asymptotic stability?

The linear case

The answer is given by the following theorem:

Theorem (Stability of linear difference equations)

Let $\rho(A) = \max\{|\lambda|, \lambda$ *is an eigenvalue of* $A\}$.

(i) $x\,(kh + h) = A \cdot x\,(kh)$ *is stable if and only* $\rho(A) \leq 1$.

(ii) $x\,(kh + h) = A \cdot x\,(kh)$ *is asymptotically stable if and only* $\rho(A) < 1$.

The linear case: intuition

1. $\|x(0) - 0\| \leq \delta \Rightarrow \|x(kh)\| \leq \varepsilon \quad ?$

   $\|x(kh)\| = \|A \cdot x((k-1)h)\| = \ldots = \left\|A^k \cdot x(0)\right\| \leq \left\|A^k\right\| \cdot \|x(0)\| \leq \|A\|^k \cdot \delta$

   To achieve stability, choose matrix A that does not grow with k

   This is when the maximum absolute eigenvalue of A, $\rho(A) \leq 1$

2. $\lim_{k\to\infty} \|x(kh)\| \leq \lim_{k\to\infty} \|A\|^k \cdot \delta \to 0$

   In this case, $\rho(A) < 1$

- In scalar case, A is constant and $\rho(A) = A$
- If the eigenvalues are larger than $1 \quad \Rightarrow$ instability

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WSNCS with constant network delay

Consider a linear model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{7}$$

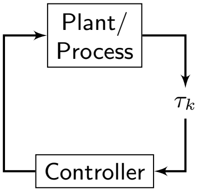
Assume that the controller takes a decision proportional to the state

$$u(t) = -Lx(t)$$

- L is chosen accordingly in order to achieve stability
- In general $u(t) = -L\hat{x}(t)$ where $\hat{x}(t)$ is an estimate of $x(t)$ based on $y(t)$
- Assume $x(t)$ is estimated perfectly

WSNCS with constant network delay

Let τ_k be the delay introduced by the network



Assume $0 \leq \tau_k \leq h$ where h the sampling time

When is the closed loop control system stable despite τ_k and given $u\left(kh\right) = -Lx\left(kh\right)$?

Network Delays

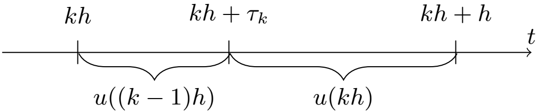
Our model becomes:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t - \tau_k), \quad t \in [kh, kh + h) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{8}$$

where $u(t)$ is the input signal in the absence of delay, i.e. $u(t) = u(kh)$ for $t \in [kh, kh + h)$.

In the presence of delay τ_k, the control command $u(hk - h)$ is used until time $t = hk + \tau_k$.

WSNCS with constant network delay



From slide 12, the solution of (7) is

$$x\left(kh+h\right) = e^{Ah}x\left(kh\right) + \int\limits_{kh}^{kh+h} e^{A(hk+h-\tau)}Bu\left(\tau\right)d\tau =$$

$$= e^{Ah}x\left(kh\right) + \int\limits_{kh}^{kh+\tau_k} e^{A(hk+h-\tau)}Bu\left(\left(k-1\right)h\right)d\tau + \int\limits_{kh+\tau_k}^{kh+h} e^{A(hk+h-\tau)}Bu\left(kh\right)d\tau =$$

by changing variable $s = kh + h - \tau$, we obtain

$$x(kh+h) = \phi x(kh) + \Gamma_0(\tau_k)u(kh) + \Gamma_1(\tau_k)u(kh-h) \tag{9}$$

$$\phi = e^{Ah} \qquad \Gamma_0\left(\tau_k\right) = \int\limits_{0}^{h-\tau_k} e^{As}Bds \qquad \Gamma_1\left(\tau_k\right) = \int\limits_{h-\tau_k}^{h} e^{As}Bds$$

Network Delays

Consider the linear control input $u(x(kh)) = -Lx(kh), \quad L \in \mathbb{R}^{n \times n}$
Then, we can write

$$x\left(kh+h\right) = \phi x\left(kh\right) - \Gamma_0\left(\tau_k\right) Lx\left(kh\right) - \Gamma_1\left(\tau_k\right) Lx\left(\left(k-1\right)h\right) \tag{10}$$

By defining the augmented state vector $z = \begin{bmatrix} x(kh) \\ x(kh-h) \end{bmatrix}$ and the matrix $\overline{\phi}(\tau_k) = \begin{bmatrix} \phi - \Gamma_0(\tau_k)L & \Gamma_1(\tau_k) \\ -L & 0 \end{bmatrix}$.

We obtain the equivalent form

$$z(kh+h) = \overline{\phi}(\tau_k)z(kh) \tag{11}$$

If the maximum eigenvalue of $\overline{\phi}$, $\rho\left(\overline{\phi}\right) < 1$, then the closed loop system is asymptotically stable.

Example

Suppose that we are given a simple scalar system that is subject to a constant network delay τ and governed by the following equation:

$$\dot{x}(t) = u(t)$$

Assume that the controller decision is $u(t) = -Lx(t)$.

In order to study the stability of the delay system, the matrix $\overline{\phi}$ needs to be constructed. Thus, since $A = 0$ and $B = 1$,

$$\phi = e^0 = 1 \qquad \Gamma_0 = \int_0^{h-\tau} ds = h - \tau \qquad \Gamma_1 = \int_{h-\tau}^{h} ds = \tau$$

and the matrix becomes

$$\overline{\phi} = \begin{bmatrix} \phi - \Gamma_0(\tau) L & \Gamma_1(\tau) \\ -L & 0 \end{bmatrix} = \begin{bmatrix} 1 - hL + \tau & \tau \\ -L & 0 \end{bmatrix}$$

Example

To compute the eigenvalues of $\overline{\phi}$, we need to solve the equation:

$$\det\left(\overline{\phi} - \lambda I\right) = 0 \Leftrightarrow (1 - hL + \tau L - \lambda)(-\lambda) + \tau L = 0 \Leftrightarrow$$

$$\Leftrightarrow \lambda^2 - \lambda(1 - hL + \tau L) + \tau L = 0 \Leftrightarrow$$

$$\Leftrightarrow \lambda_{1,2} = \frac{1 - hL + \tau L \pm \sqrt{(1 - hL + \tau L)^2 - 4\tau L}}{2}$$

The sampled system state is asymptotically stable iff $|\lambda_1|, |\lambda_2| < 1$.

Summary

- We have seen the basic aspects of control systems
  - Mathematical description of the state evolution
  - Discretization
  - Stability

Next lecture

- WSNCS, robustness to packet delays and losses