Principles of Wireless Sensor Networks

https://kth.instructure.com/courses/2912/

Lecture 8

Static Distributed Estimation

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Course content

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- Part 2
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  - Lec 3: Physical Layer
  - Lec 4: Medium Access Control Layer
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  - Lec 11: Time Synchronization
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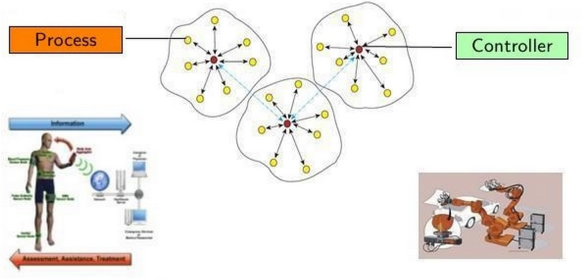
Today's lecture



- Today we study how to perform static estimation from noisy measurements of the sensors
- "Static" means that the estimation is of a variable (constant or random) that does not evolve over time

Motivation for Static Estimation

- Plays a central role in many WSNs applications
- Accurately predicts the **parameters** of a **phenomenon**
- **Communication:** position, navigation
- **Monitoring:** pollution, earthquake magnitude
- **Surveillance:** crowd density, intruders, attitude

Today's learning goals

- What are the **fundamental aspects** of distributed estimation?
- Estimation over a Star and a General topology?
- What is the LMMSE estimator?
- How to make a static sensor fusion?

Outline

- Star and general topologies
- Estimation from one sensor
- Distributed estimation in a star topology
- Distributed estimation in a general topology

Outline

- Star and general topology
- Estimation from one sensor
  - Model of the measurements for one sensor
  - Model of the estimator
  - Mean Squared Error (MSE)
  - LMMSE estimate
- Distributed estimation from many sensors
  - Star topology
  - General topology

Topology 1: Star topology

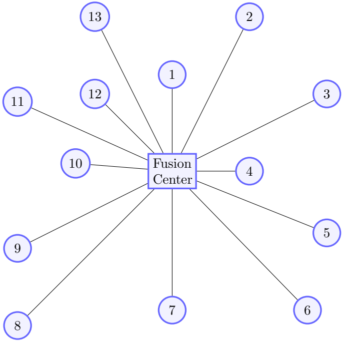


Figure: Network with a star topology: Solid lines indicating that there is message communication between nodes. The fusion center receives information from all other nodes.

- The **phenomenon** is observed by a number of sensors organized as a star
- Multiple sensors make measurements
- Measurements are transmitted to a fusion center

Topology 2: General topology

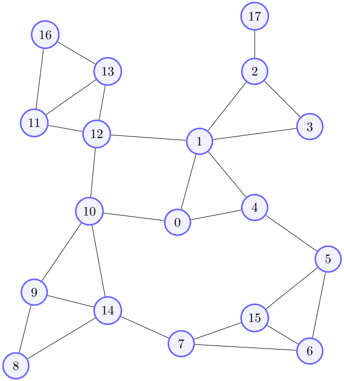


Figure: Network with an Arbitrary Topology: Solid lines indicating that there is communication between nodes. There is no node acting as fusion center.

- The **phenomenon** is observed by a number of sensors organized arbitrarily
- Multiple sensors make measurements
- Measurements are not transmitted to a fusion center
  - Indeed, no fusion center. Every node is a sort of local fusion center

Outline

- Star and General topology
- Estimation from one sensor
  - Model of the measurements for one sensor
  - Model of the estimator
  - Mean Squared Error (MSE)
  - LMMSE estimate
- Distributed estimation from many sensors
  - Star topology
  - General topology

Model of the measurements for one sensor

- Let's consider only one sensor
- **Linear** measurements (i.e., measurements and the parameters are related linearly) with **noise** or **measurement errors**

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v} \tag{1}$$

- $\mathbf{y}$: sensor measurement(s)
- $\mathbf{H}$: a known matrix
- $\mathbf{x}$: what we want to estimate
- $\mathbf{v}$: unknown noise or measurement error

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- Let's consider only one sensor
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- $\mathbf{y}$: sensor measurement(s)
- $\mathbf{H}$: a known matrix
- $\mathbf{x}$: what we want to estimate
- $\mathbf{v}$: unknown noise or measurement error
- Goal: How to estimate **x** out of the measurement **y**?

Model of the estimator

Linear estimator, i.e., the estimator and the measurements are assumed to be linearly related

$$\hat{\mathbf{x}}(\mathbf{L}) = \mathbf{L}\mathbf{y}$$

- $\mathbf{y}$: sensor measurement(s)
- $\hat{\mathbf{x}}(\mathbf{L})$: estimator of $\mathbf{x}$, dependent on $\mathbf{L}$
- We need to compute a good estimate $\hat{\mathbf{x}}(\cdot) \Rightarrow$ what matrix $\mathbf{L}$ to be used?
- **Performance criterion** for computing $\mathbf{L}$?

Mean Squared Error (MSE)

A good estimate $\hat{\mathbf{x}}(\cdot)$ is found by considering the **MSE**, which is given by the trace of **error covariance matrix** $\mathbf{C}$ of the estimator error

- In particular, for fixed $\mathbf{L}$, MSE is defined as

$$\begin{aligned} \mathrm{MSE}(\mathbf{L}) &= \mathsf{Tr}\left\{\mathbf{C}(\mathbf{L})\right\} \\ &= \mathsf{Tr}\left\{\mathsf{E}\left\{(\hat{\mathbf{x}}(\mathbf{L}) - \mathbf{x})\,(\hat{\mathbf{x}}(\mathbf{L}) - \mathbf{x})^{\mathsf{T}}\right\}\right\} \\ &= \textstyle\sum_{i=1}^{N} \mathsf{E}(\hat{x}_i(\mathbf{L}) - x_i)^2 \end{aligned}$$

- Let $\mathbf{L}^\star = \arg\min_{\mathbf{L}} \mathrm{MSE}(\mathbf{L})$

- Then, $\hat{\mathbf{x}} = \mathbf{L}^\star \mathbf{y}$ is called the **linear minimum MSE (LMMSE)** estimate of $\mathbf{x}$

LMMSE estimate

Proposition 1

Consider a random variable $\mathbf{x}$ *being observed by a sensor that generates measurements of the form* $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$. *Then LMMSE estimator of* $\mathbf{x}$ *given* $\mathbf{y}$ *is*

$$\hat{\mathbf{x}} = \underbrace{\mathbf{P}\mathbf{H}^T\mathbf{R}_{\mathbf{v}}^{-1}}_{\mathbf{L}^\star}\mathbf{y} \ ,$$

where

$$\mathbf{P} = \left(\mathbf{R}_{\mathbf{x}}^{-1} + \mathbf{H}^T\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}\right)^{-1} \ ,$$

$\mathbf{R}_{\mathbf{x}}$ is the covariance matrix of $\mathbf{x}$, and $\mathbf{R}_{\mathbf{v}}$ is the noise covariance matrix.

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- We need to show that $\mathbf{L}^\star = \mathbf{P}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}$

LMMSE estimate proof

Advanced topic, not requested for the exam

LMMSE estimate proof

Advanced topic, not requested for the exam

Proof:

Preliminaries:

$$
\begin{aligned}
&(1) \quad \mathbf{A} + \mathbf{B} \succeq \mathbf{B} \text{ when } \mathbf{A} \succeq \mathbf{0} \\
&(2) \quad \mathbf{A} \succeq \mathbf{B} \Rightarrow \mathrm{Tr}(\mathbf{A}) \geq \mathrm{Tr}(\mathbf{B}) \\
&(3) \quad (\mathbf{A} + \mathbf{B}\mathbf{C})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{I} + \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1}
\end{aligned}
$$

LMMSE estimate proof

Proof:

$$
\begin{aligned}
\mathbf{C}(\mathbf{L}) &= \mathsf{E}\left\{(\hat{\mathbf{x}}(\mathbf{L})-\mathbf{x})\,(\hat{\mathbf{x}}(\mathbf{L})-\mathbf{x})^{\mathsf{T}}\right\} = \mathsf{E}\left\{(\mathbf{L}\mathbf{y}-\mathbf{x})\,(\mathbf{L}\mathbf{y}-\mathbf{x})^{\mathsf{T}}\right\} \\
&= \mathsf{E}\left\{(\mathbf{L}\mathbf{H}-\mathbf{I})\,\mathbf{x}\mathbf{x}^{\mathsf{T}}\,(\mathbf{L}\mathbf{H}-\mathbf{I})^{\mathsf{T}} + \mathbf{L}\mathbf{v}\mathbf{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} - \mathbf{L}\mathbf{H}\mathbf{x}\mathbf{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} - \mathbf{L}\mathbf{v}\mathbf{x}^{\mathsf{T}}\mathbf{H}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}}\right\} \\
&= (\mathbf{L}\mathbf{H}-\mathbf{I})\,\mathbf{R}_{\mathbf{x}}\,(\mathbf{L}\mathbf{H}-\mathbf{I})^{\mathsf{T}} + \mathbf{L}\mathbf{R}_{\mathbf{v}}\mathbf{L}^{\mathsf{T}} - \mathbf{L}\mathbf{H}\,\underbrace{\mathsf{E}\{\mathbf{x}\mathbf{v}^{\mathsf{T}}\}}_{0}\,\mathbf{L}^{\mathsf{T}} - \mathbf{L}\,\underbrace{\mathsf{E}\{\mathbf{v}\mathbf{x}^{\mathsf{T}}\}}_{0}\,\mathbf{H}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}}
\end{aligned}
$$

LMMSE estimate proof

Proof:

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&= \mathsf{E}\left\{(\mathbf{L}\mathbf{H}-\mathbf{I})\mathbf{x}\mathbf{x}^{\mathsf{T}}(\mathbf{L}\mathbf{H}-\mathbf{I})^{\mathsf{T}} + \mathbf{L}\mathbf{v}\mathbf{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} - \mathbf{L}\mathbf{H}\mathbf{x}\mathbf{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} - \mathbf{L}\mathbf{v}\mathbf{x}^{\mathsf{T}}\mathbf{H}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}}\right\} \\
&= (\mathbf{L}\mathbf{H}-\mathbf{I})\mathbf{R}_{\mathbf{x}}(\mathbf{L}\mathbf{H}-\mathbf{I})^{\mathsf{T}} + \mathbf{L}\mathbf{R}_{\mathbf{v}}\mathbf{L}^{\mathsf{T}} - \mathbf{L}\mathbf{H}\underbrace{\mathsf{E}\{\mathbf{x}\mathbf{v}^{\mathsf{T}}\}}_{0}\mathbf{L}^{\mathsf{T}} - \mathbf{L}\underbrace{\mathsf{E}\{\mathbf{v}\mathbf{x}^{\mathsf{T}}\}}_{0}\mathbf{H}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} \\
&= \mathbf{L}\underbrace{\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_{\mathbf{v}}\right)}_{\mathbf{S}}\mathbf{L}^{\mathsf{T}} - \mathbf{L}\mathbf{H}\mathbf{R}_{\mathbf{x}} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} + \mathbf{R}_{\mathbf{x}} \\
&= \mathbf{L}\mathbf{S}\mathbf{L}^{\mathsf{T}} - \mathbf{L}(\mathbf{S}\mathbf{S}^{-1})\mathbf{H}\mathbf{R}_{\mathbf{x}} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}(\mathbf{S}\mathbf{S}^{-1})\mathbf{L}^{\mathsf{T}} + \mathbf{R}_{\mathbf{x}} \\
&\qquad + \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{x}} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{x}} =
\end{aligned}
$$

LMMSE estimate proof

Proof:

$$\begin{aligned}
\mathbf{C}(\mathbf{L}) &= \mathsf{E}\left\{(\hat{\mathbf{x}}(\mathbf{L})-\mathbf{x})\,(\hat{\mathbf{x}}(\mathbf{L})-\mathbf{x})^{\mathsf{T}}\right\} = \mathsf{E}\left\{(\mathbf{Ly}-\mathbf{x})\,(\mathbf{Ly}-\mathbf{x})^{\mathsf{T}}\right\} \\
&= \mathsf{E}\left\{(\mathbf{LH}-\mathbf{I})\,\mathbf{xx}^{\mathsf{T}}\,(\mathbf{LH}-\mathbf{I})^{\mathsf{T}} + \mathbf{Lvv}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} - \mathbf{LHxv}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} - \mathbf{Lvx}^{\mathsf{T}}\mathbf{H}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}}\right\} \\
&= (\mathbf{LH}-\mathbf{I})\,\mathbf{R_x}\,(\mathbf{LH}-\mathbf{I})^{\mathsf{T}} + \mathbf{LR_vL}^{\mathsf{T}} - \mathbf{LH}\,\underbrace{\mathsf{E}\{\mathbf{xv}^{\mathsf{T}}\}}_{0}\,\mathbf{L}^{\mathsf{T}} - \mathbf{L}\,\underbrace{\mathsf{E}\{\mathbf{vx}^{\mathsf{T}}\}}_{0}\,\mathbf{H}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} \\
&= \mathbf{L}\underbrace{\left(\mathbf{HR_xH}^{\mathsf{T}} + \mathbf{R_v}\right)}_{\mathbf{S}}\mathbf{L}^{\mathsf{T}} - \mathbf{LHR_x} - \mathbf{R_xH}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} + \mathbf{R_x} \\
&= \mathbf{LSL}^{\mathsf{T}} - \mathbf{L}(\mathbf{SS}^{-1})\mathbf{HR_x} - \mathbf{R_xH}^{\mathsf{T}}(\mathbf{SS}^{-1})\mathbf{L}^{\mathsf{T}} + \mathbf{R_x} \\
&\qquad + \mathbf{R_xH}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{H}^{\mathsf{T}}\mathbf{R_x} - \mathbf{R_xH}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{H}^{\mathsf{T}}\mathbf{R_x} = \\
&= \left(\mathbf{L}-\mathbf{R_xH}^{\mathsf{T}}\mathbf{S}^{-1}\right)\mathbf{S}\left(\mathbf{L}-\mathbf{R_xH}^{\mathsf{T}}\mathbf{S}^{-1}\right)^{\mathsf{T}} + \mathbf{R_x} - \mathbf{R_xH}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{H}^{\mathsf{T}}\mathbf{R_x}
\end{aligned}$$

LMMSE estimate proof

Proof:

$$
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&\succeq \mathbf{R}_{\mathbf{x}} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{x}}
\end{aligned}
$$

LMMSE estimate proof

The lower bound is achieved when

$$\begin{aligned}
\mathbf{L} &= \mathbf{R_x}\mathbf{H}^\mathsf{T}\mathbf{S}^{-1} = \mathbf{R_x}\mathbf{H}^\mathsf{T}\left(\mathbf{H}\mathbf{R_x}\mathbf{H}^\mathsf{T}+\mathbf{R_v}\right)^{-1} \\
&= \mathbf{R_x}\mathbf{H}^\mathsf{T}\left(\mathbf{R_v}^{-1} - \mathbf{R_v}^{-1}\mathbf{H}\left(\mathbf{I}+\mathbf{R_x}\mathbf{H}^\mathsf{T}\mathbf{R_v}^{-1}\mathbf{H}\right)^{-1}\mathbf{R_x}\mathbf{H}^\mathsf{T}\mathbf{R_v}^{-1}\right) \\
&= \left(\mathbf{I} - \mathbf{R_x}\mathbf{H}^\mathsf{T}\mathbf{R_v}^{-1}\mathbf{H}\left(\mathbf{I}+\mathbf{R_x}\mathbf{H}^\mathsf{T}\mathbf{R_v}^{-1}\mathbf{H}\right)^{-1}\right)\mathbf{R_x}\mathbf{H}^\mathsf{T}\mathbf{R_v}^{-1} \\
&= \left(\mathbf{I}+\mathbf{R_x}\mathbf{H}^\mathsf{T}\mathbf{R_v}^{-1}\mathbf{H}\right)^{-1}\mathbf{R_x}\mathbf{H}^\mathsf{T}\mathbf{R_v}^{-1} = \left(\mathbf{R_x}^{-1}+\mathbf{H}^\mathsf{T}\mathbf{R_v}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^\mathsf{T}\mathbf{R_v}^{-1} = \mathbf{P}\mathbf{H}^\mathsf{T}\mathbf{R_v}^{-1} \ \square
\end{aligned}$$

LMMSE estimate

Recap:

Consider the linear system of measurements given in (1), i.e., $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$. Let $\hat{\mathbf{x}}$ denote the LMMSE estimator of $\mathbf{x}$ given $\mathbf{y}$. Then we have

$$\mathbf{P}^{-1}\hat{\mathbf{x}} = \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{y} \,, \tag{2}$$

where

$$\mathbf{P} = \left(\mathbf{R}_{\mathbf{x}}^{-1} + \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}\right)^{-1} = \text{error covariance of } \hat{\mathbf{x}}.$$

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- Relation (2) has been derived for **the case of one sensor**
- In **the case of multiple sensors**, relation (2) suggests the possibility of **combining local estimates directly**

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- In **the case of multiple sensors**, relation (2) suggests the possibility of **combining local estimates directly**
- Several measurements from **one sensor** can be seen in case of **multiple sensors**

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- In **the case of multiple sensors**, relation (2) suggests the possibility of **combining local estimates directly**
- Several measurements from **one sensor** can be seen in case of **multiple sensors**
- **No** need to send all the measurements to a **central data processing**
- This is called **static sensor fusion**

Some considerations on $\mathbf{x}$

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- When **no prior information** is available, then $\mathbf{R_x^{-1}} = 0$ and $\mathbf{P} = \left(\mathbf{H}^\mathsf{T}\mathbf{R_v^{-1}}\mathbf{H}\right)^{-1}$, thus giving

$$\hat{\mathbf{x}} = \left(\mathbf{H}^\mathsf{T}\mathbf{R_v^{-1}}\mathbf{H}\right)^{-1}\mathbf{H}^\mathsf{T}\mathbf{R_v^{-1}}\mathbf{y}$$

which is also denoted as the **weighted least square estimate**.

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  which is also denoted as the **weighted least square estimate**.

- If the information included has a **non zero mean**, the estimate need to be corrected in the following way

$$\mathbf{P}^{-1}(\hat{\mathbf{x}} - \bar{\mathbf{x}}) = \mathbf{H}^\mathsf{T}\mathbf{R_v}^{-1}(\mathbf{y} - H\bar{\mathbf{x}})$$

- We assume $\bar{\mathbf{x}} = 0$ for readability reasons.

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- Star and General topology
- Estimation from one sensor
  - Model of the measurements for one sensor
  - Model of the estimator
  - Mean Squared Error (MSE)
  - LMMSE estimate
- Distributed estimation from many sensors
  - Star topology
  - General topology

Static sensor fusion, star topology

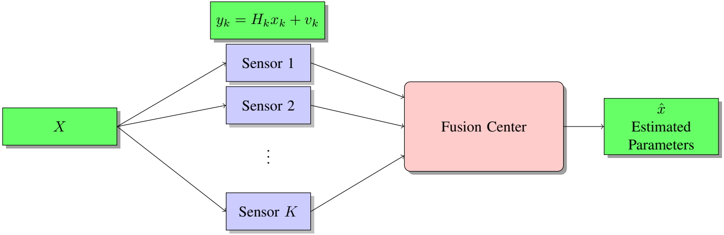

Figure: Illustration of how the process in static sensor fusion is preformed.

- Now we move to a case of **many sensors in a star topology**

Static sensor fusion, star topology

Proposition 2

Consider a random variable $\mathbf{x}$ *being observed by* K *sensors that generate measurements of the form*

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x} + \mathbf{v}_k, \quad k = 1, \dots, K$$

where the $\mathbf{v}_k$ *and* $\mathbf{v}_j$ $(j \neq k)$ *are uncorrelated.*

Static sensor fusion, star topology

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where the $\mathbf{v}_k$ *and* $\mathbf{v}_j$ $(j \neq k)$ *are uncorrelated.*
Let $\hat{\mathbf{x}}$ *denote the LMMSE estimator of* $\mathbf{x}$ *given* $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_K)$*, as obtained at the fusion center. Then*

$$\mathbf{P}^{-1}\hat{\mathbf{x}} = \sum_{k=1}^{K} \mathbf{P}_k^{-1}\hat{\mathbf{x}}_k \ ,$$

where $\mathbf{P}$ *is the estimate error covariance corresponding to* $\hat{\mathbf{x}}$ *and* $\mathbf{P}_k$ *is the error covariance corresponding to* $\hat{\mathbf{x}}_k$*.*

Static sensor fusion, star topology

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$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x} + \mathbf{v}_k, \quad k = 1, \ldots, K$$

where the $\mathbf{v}_k$ *and* $\mathbf{v}_j$ $(j \neq k)$ *are uncorrelated.*
Let $\hat{\mathbf{x}}$ *denote the LMMSE estimator of* $\mathbf{x}$ *given* $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_K)$*, as obtained at the fusion center. Then*

$$\mathbf{P}^{-1}\hat{\mathbf{x}} = \sum_{k=1}^{K} \mathbf{P}_k^{-1}\hat{\mathbf{x}}_k \ ,$$

where $\mathbf{P}$ *is the estimate error covariance corresponding to* $\hat{\mathbf{x}}$ *and* $\mathbf{P}_k$ *is the error covariance corresponding to* $\hat{\mathbf{x}}_k$*. Furthermore,*

$$\mathbf{P}^{-1} = -(K-1)\mathbf{R}_{\mathbf{x}}^{-1} + \sum_{k=1}^{K} \mathbf{P}_k^{-1} \ ,$$

$\mathbf{R}_{\mathbf{x}}$ *is the covariance matrix of* $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_K)$

Proof of proposition 2

Proof: Note that overall linear system is given by

$$\underbrace{\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_K \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \mathbf{H}_1 \\ \vdots \\ \mathbf{H}_K \end{bmatrix}}_{\mathbf{H}} \mathbf{x} + \underbrace{\begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_K \end{bmatrix}}_{\mathbf{v}}$$

Now use Proposition 1

$$\mathbf{P}^{-1}\hat{\mathbf{x}} = \mathbf{H}^\mathsf{T}\mathbf{R}_\mathbf{v}^{-1}\mathbf{y} = \begin{bmatrix} \mathbf{H}_1^\mathsf{T} \cdots \mathbf{H}_K^\mathsf{T} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{\mathbf{v}_1}^{-1} & 0 & \cdots & 0 \\ 0 & \mathbf{R}_{\mathbf{v}_2}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{R}_{\mathbf{v}_K}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_K \end{bmatrix}$$

$$= \sum_{k=1}^{K} \mathbf{H}_k^\mathsf{T}\mathbf{R}_{\mathbf{v}_k}^{-1}\mathbf{y}_k$$

$$= \sum_{k=1}^{K} \mathbf{P}_k^{-1}\hat{\mathbf{x}}_k$$

Proof of proposition 2

Moreover, from Proposition 1

$$\begin{aligned}\mathbf{P}^{-1} &= \mathbf{R}_{\mathbf{x}}^{-1} + \underbrace{\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}} \\ &= \mathbf{R}_{\mathbf{x}}^{-1} + \textstyle\sum_{k=1}^{K} \underbrace{\mathbf{H}_k^{\mathsf{T}}\mathbf{R}_{\mathbf{v}_k}^{-1}\mathbf{H}_k} \\ &= \mathbf{R}_{\mathbf{x}}^{-1} + \textstyle\sum_{k=1}^{K} \left(\mathbf{P}_k^{-1} - \mathbf{R}_{\mathbf{x}}^{-1}\right) = -(K-1)\mathbf{R}_{\mathbf{x}}^{-1} + \textstyle\sum_{k=1}^{K} \mathbf{P}_k^{-1} \ ,\end{aligned}$$

Static sensor fusion from multiple sensors

- By Proposition 2, **complexity** of the **fusion center goes down** considerably
- **Some computational load is delegated** to the distributed **sensors**
- **Each estimate is weighted** by the inverse of the error covariance matrix
- The **higher the confidence** we have in a particular sensor, the **higher the trust** we place in its measurement

Static sensor fusion from multiple sensors

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- **Each estimate is weighted** by the inverse of the error covariance matrix
- The **higher the confidence** we have in a particular sensor, the **higher the trust** we place in its measurement
- **Two step procedure**
  - All the nodes transmit local estimates to a central node (**called fusion center**)
  - Central node calculates and transmits the weighted sum of the local estimates back
- Final outcome is a **weighted average**

Outline

- Star and General topology
- Estimation from one sensor
  - Model of the measurements for one sensor
  - Model of the estimator
  - Mean Squared Error (MSE)
  - LMMSE estimate
- Distributed estimation from many sensors
  - Star topology
  - General topology

Network with arbitrary topology

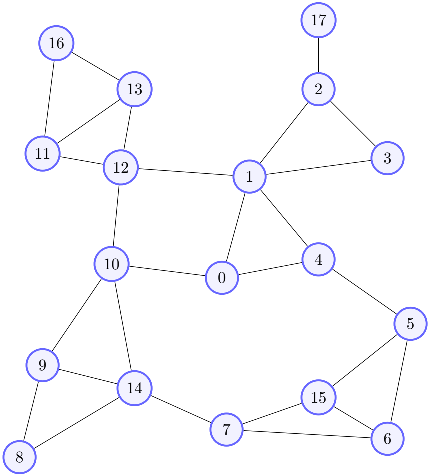

Figure: Network with a Arbitrary Topology: Solid lines indicating that there is message communication between nodes. There is no node acting as fusion center.

Network with arbitrary topology

- Generalize the static sensor fusion approach to an **arbitrary graph**
- This approaches are along the lines of **average consensus algorithms**
- **No fusion center**

Static sensor fusion with limited communication range

Example scenario:

- K nodes measure a **scalar** value x, measurements are noisy
- Nodes are connected according to an arbitrary graph
- Each node wants to calculate the average of all the scalars

$$y_k = x + v_k \ , \ k = 1, \ldots, K$$

Static sensor fusion with limited communication range

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Remember: Provided the noise components are iid Gaussian, then the maximum likelihood **(ML)** estimate $\hat{x}$ of x is given by the average of all y_k values, i.e.,

$$\hat{x} = (1/K) \sum_{k=1}^{K} y_k = (1/K)\mathbf{1}^\mathsf{T}\mathbf{y}$$

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Question: How to obtain $\hat{x}$ just by coordinating with **adjacent neighbors** (no central fusion center)?

Static sensor fusion with limited communication range

One way:

- Iterative method, iterations $n = 0, 1, 2, \ldots$
- Each sensor k, during iteration 0, set $x_{0,k} = y_k$
- Each sensor k implements the dynamical system

$$x_{n+1,k} = x_{n,k} + h \sum_{j \in \mathcal{N}_k} (x_{n,j} - x_{n,k}) \ ,$$

  where $\mathcal{N}_k$ is the adjacent sensors of sensor k
- Just **local communications**

Static sensor fusion with limited communication range

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- **Compact form**

$$\mathbf{x}_{n+1} = (\mathbf{I} - h\mathbf{L})\mathbf{x}_n \ , \ n = 0, 1, 2, \ldots \ ,$$

  where $\mathbf{L}$ is the **Graph Laplacian matrix**

Static sensor fusion with limited communication range

If the underlying graph is connected (i.e., there is at least one path among all pairs of nodes), then the **Graph Laplacian matrix** $\mathbf{L}$ has the following properties:

- $\mathbf{L}$ is symmetric positive-definite matrix.
- Each row sum of $\mathbf{L}$ is 0.
- Each column sum of $\mathbf{L}$ is 0.

Then, given small h, it can be proved that the iteration always converges to the equilibrium $(\mathbf{x}_{n+1})_k = \hat{x}$ for all $k = 1, \ldots, K$.

Static sensor fusion with limited communication range

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The idea extends in a straightforward manner to more general models such as

$$x_{n+1,k} = x_{n,k} + h\mathbf{W}_k^{-1} \sum_{j \in \mathcal{N}_k} (x_{n,j} - x_{n,k})$$

Summary

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  - Model of the estimator
  - Mean Squared Error (MSE)
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Next lecture

- Dynamic distributed estimation