Infinite Series Representation of the Bivariate K Distribution with Arbitrary Fading Parameters

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***Abstract*—The bivariate K-distribution with arbitrary correlation and not necessarily identical distributed (id) parameters is introduced and analyzed. Novel analytical expressions for the bivariate probability density and the joint moments are derived for the general case where the envelopes and their powers have different correlation coefficients. For independent powers, the cumulative distribution, and characteristic functions are analytically obtained. As an application example, the bivariate K-distribution is applied to fading channels modeling and the outage performance of dual selection diversity receivers is assessed.**

***Index Terms*—Correlated-K distribution, bivariate statistics, composite fading/shadowing channels, radar clutter modeling.**

I. Introduction

SEVERAL mixed channel models have been proposed in the research literature in order to simultaneously take into account both short- and long-term fading conditions. The most popular among these fading models encompass the Suzuki, Nakagami-lognormal, and Rice-lognormal models [1]. A recently applied mixture fading model is the K distribution [1], [2], which arises in situations where the envelope of the signal is Rayleigh distributed with average power being gamma distributed. The K- distribution has been extensively used for modeling electromagnetic scattering from physical media such as tropospheric propagation of radio waves, various types of radar clutter, and phenomenological description of sea clutter [3]–[5]. More specifically, the K-distribution is considered to be appropriate for modeling both sea and land clutters. In [4] the joint probability density function (PDF) and the moments of two clutter amplitudes following the K-distribution were derived, assuming identical scaling and shaping parameters. An extension to this work was provided in [5] where correlation at the power of the clutter amplitudes was also considered, with identical parameters. In the context of wireless communications over fading channels, several bivariate distributions have been applied in the past, including the Rayleigh, Nakagami-m, Weibull, and the generalized gamma [1], [6]–[8]. However, to the best of the authors' knowledge, no previous considered model incorporates the correlated small- and large-scale effects.

Motivated by the preceedings in this letter, novel analytical expressions for the bivariate K distribution, with not necessarily identically distributed (id) parameters, and arbitrary correlation in both the envelopes and the powers, are presented. These expressions include the joint PDF, cumulative distribution function (CDF), and the characteristic function (CF). Based on these expressions, the outage probability at the output of dual-branch selection combining (SC) receivers operating over correlated K fading channels is derived.

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II. Channel Model

The K-distribution is a mixture of a zero mean complex Gaussian and a gamma distribution. Let R_i $(i = 1$ and $2)$ denote the envelopes of two zero mean complex Gaussian random variables (RVs). Then, R_1 and R_2 are Rayleigh RVs with joint PDF given by

$$f_{R_1,R_2}(x_1,x_2) = \sum_{t=0}^{\infty} \frac{4\rho_R^t/\left(t!\right)^2}{\left(1-\rho_R\right)^{2t+1}} \times \prod_{i=1}^{2} \frac{x_i^{2t+1}}{G_i^{t+1}} \exp\left[-\frac{x_i^2}{(1-\rho_R)G_i}\right] \tag{1}$$

with $G_i = \mathbb{E}_{R_i}\langle x_i^2\rangle$ and $\mathbb{E}_{R_i}\langle\cdot\rangle$ denoting averaging over the distribution of R_i, and ρ_R is the power correlation coefficient, i.e., between R_1^2 and R_2^2. Assume that the average powers, G_1 and G_2, become random and correlation exists between them. Their joint gamma PDF, can be derived using [6, eq. (12)] as

$$f_{G_1,G_2}\left(x_1,x_2\right) = \sum_{h=0}^{\infty} \frac{(k_1)_h \rho_G^h}{h!(1-\rho_G)^{k_1+2h}} \times {}_1F_1\left[k_2-k_1; k_2+h; \frac{\rho_G x_2}{(1-\rho_G)\Omega_2}\right] \times \prod_{i=1}^{2} \frac{x_i^{k_i+h-1}\exp\left[-x_i/\left[(1-\rho_G)\Omega_i\right]\right]}{\Gamma(k_i+h)\Omega_i^{k_i+h}} \tag{2}$$

where $k_i > 0$ is the shaping parameter, $\Omega_i = \mathbb{E}_{G_i}\langle x_i^2\rangle$, ρ_G is the power correlation coefficient between G_1 and G_2, ${}_1F_1(\cdot)$ is the confluent hypergeometric function [9, eq. (9.21)], $(\alpha)_n$ is the Pochhammer symbol, and $\Gamma(\cdot)$ is the Gamma function [9, eq. (8.310/1)].

The correlated K-distribution can be obtained by averaging the G_i of (1) over the conditional density function of the bivariate gamma (2). This can be mathematically expressed as

$$f_{X_1,X_2}(x,y) = \int_0^\infty \int_0^\infty f_{R_1|G_1,R_2|G_2}(x|v,y|w) \times f_{G_1,G_2}(v,w)dvdw. \tag{3}$$

Substituting (1) and (2) in (3), integrals of the following form need to be solved $\mathcal{I} = \int_0^\infty y^\alpha \exp\left(-A_1 y^{-1} - A_2 y\right) {}_1F_1(B_1; B_2; \rho_G\, A_2\, y)dy$. These integrals can be solved by using the infinite series representation of ${}_1F_1(\cdot)$ [9, eq. (9.21/1)] and then using [9, eq. (3.471/9)]. Hence, the joint PDF of the K-distribution with arbitrary correlation and not necessarily id statistical parameters can be obtained as

$$f_{X_1,X_2}(x_1,x_2) = \sum_{t,h,q=0}^{\infty} \frac{16(k_1)_h \rho_G^{h+q} \rho_R^t (k_2-k_1)_q}{h!q!(t!)^2 \Gamma(k_1+h)\Gamma(k_2+h+q)} \times \frac{1}{(1-\rho_G)^{1-k_2}} \prod_{i=1}^{2} \frac{x_i^{\tau_i} \Omega_i^{-(\tau_i+1)/2}}{\left[(1-\rho_R)(1-\rho_G)\right]^{\tau_i/2}} \times K_{\tau_i-2t-1}\left[\frac{2x_i/\sqrt{\Omega_i}}{\sqrt{(1-\rho_R)(1-\rho_G)}}\right] \tag{4}$$

where $K_v(\cdot)$ is the second kind modified Bessel function of order v [9, eq. (8.407/1)] and $\tau_i = k_i + h + t + (i-1)q$. Setting $\Omega_1 = \Omega_2$ and $k_1 = k_2$ in (4), it simplifies to previous known result [5, eq. (29)]. Under the assumption that no correlation between G_1 and G_2 exists, i.e., $\rho_G = 0$, (4) simplifies to

$$f_{X_1,X_2}(x_1,x_2) = \sum_{t=0}^{\infty} \frac{16\rho_R^t/(t!)^2}{(1-\rho_R)^{(k_1+k_2)/2+t}} \times \prod_{i=1}^{2} \frac{x_i^{k_i+t}/\Gamma(k_i)}{\Omega_i^{(k_i+t+1)/2}} K_{k_i-t-1}\left[\frac{2x_i}{\sqrt{1-\rho_R \Omega_i}}\right]. \tag{5}$$

Setting $\Omega_1 = \Omega_2$ and $k_1 = k_2$, (5) further simplifies to previous known results [4, eq (12)], i.e., the bivariate K PDF with id parameters. Starting from (4) and using [9, eqs. (6.561/16) and (9.14/1)], the joint moments $\mu(n_1,n_2) = \mathbb{E}\langle x_1^{n_1} x_2^{n_2}\rangle$ can be obtained as

$$\mu(n_1,n_2) = \left[\prod_{i=1}^{2} (1-\rho_G)^{(n_i+k_2)/2} \Omega_i^{n_i/2} \Gamma\left(\frac{n_i}{2}+1\right)\right] \times \frac{\Gamma(k_1+n_1/2)}{\Gamma(k_1)} {}_2F_1\left(-\frac{n_1}{2}, -\frac{n_2}{2}; 1; \rho_R\right) \sum_{q=0}^{\infty} \frac{(k_2-k_1)_q}{\Gamma(k_2+q)q!} \times \Gamma(k_2 + \frac{n_2}{2} + q)\rho_G^q {}_2F_1\left(k_1 + \frac{n_1}{2}, k_2 + \frac{n_2}{2}; k_2+q; \rho_G\right) \tag{6}$$

with ${}_2F_1(\cdot)$ being the Gauss hypergeometric function [9, eq. (9.100)]. Assuming $k = k_1 = k_2$, (6) can be obtained in closed form as

$$\mu_{X_1,X_2}(n_1,n_2) = \left[\prod_{i=1}^{2} \frac{\Gamma\left(n_i/2+1\right)\Gamma\left(k+n_i/2\right)\Omega_i^{n_i/2}}{\Gamma(k)}\right] \times {}_2F_1\left(-\frac{n_1}{2}, -\frac{n_2}{2}; 1; \rho_N\right) {}_2F_1\left(-\frac{n_1}{2}, -\frac{n_2}{2}; k; \rho_G\right). \tag{7}$$

The power correlation coefficient between X_1^2 and X_2^2 is defined as $\rho \triangleq C_{X_1^2,X_2^2} \Big/ \left(\sigma_{X_1^2}\sigma_{X_2^2}\right)$ [10, eq. (7.7)], where $C_{X_1^2,X_2^2}$ is the covariance and $\sigma_{X_i^2}$ is the standard deviation. Hence, substituting (7) and [1, eq. (64)] in this definition, ρ can be obtained as follows

$$\rho = \frac{(k+\rho_G)(1+\rho_R)-k}{k+2}. \tag{8}$$

Moreover, using (5) and [9, eq. (6.621/3)], the joint CF of X_1 and X_2 can be expressed as

$$\Phi_{X_1,X_2}(s_1,s_2) = (1-\rho_R)\pi \left[\prod_{i=1}^{2} \frac{\Gamma\left(k_i+\frac{1}{2}\right)}{\Gamma(k_i)}\right] \sum_{t=0}^{\infty} \rho_R^t \times \left[\frac{[(2t+1)!/t!]}{2^{2t-(k_1+k_2)-1}}\right]^2 \prod_{i=1}^{2} \frac{1/\Gamma(k_i+t+3/2)}{\left[s_i\sqrt{(1-\rho_R)\Omega_i}+2\right]^{2k_i}} \times {}_2F_1\left[2k_i, k_i-t-\frac{1}{2}; k_i+t+\frac{3}{2}; \frac{s_i\sqrt{(1-\rho_R)\Omega_i}-2}{s_i\sqrt{(1-\rho_R)\Omega_i}+2}\right]. \tag{9}$$

Also, the joint CDF of X_1 and X_2 can be obtained from (5) and [11, eq (03.04.21.0007.01)] as

$$F_{X_1,X_2}(x_1,x_2) = \pi^2(1-\rho_R) \sum_{t=0}^{\infty} \rho_R^t \prod_{i=1}^{2} \csc\left[\pi\left(k_i-t+1\right)\right] \times \left\{ \frac{\xi_i^{t+1}}{\Gamma(k_i)} {}_1\tilde{F}_2\left[t+1; t+2-k_i, t+2; \xi_i\right] - \frac{\xi^{k_i}}{t!} {}_1\tilde{F}_2\left[k_i; k_i-t, k_i+1; \xi_i\right] \right\} \tag{10}$$

where $\xi_i = x_i^2/\left[(1-\rho_R)\Omega_i\right]$ and ${}_1\tilde{F}_2(\cdot)$ is the regularized generalized hypergeometric function [11, eq. (07.32.02.0001.01)].

III. SC Receiver Outage Probability

As a representative example of the applicability of the previous derived results, the bivariate K-distribution is considered as a fading channel model. Let us consider a dual branch SC diversity receiver operating over a flat fading/shadowing channel, modeled by the K-distribution, plus additive white Gaussian noise with power spectral density N_0. Let γ_i denoting the instantaneous SNR of the i-th input branch. The joint CDF of γ_1 and γ_1, $F_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2)$, can be extracted by performing RVs transformations of the form $\gamma_i = x_i^2 E_s/N_0$ in (10), where E_s is the transmitted signal energy. The corresponding average value of γ_i can be expressed as $\overline{\gamma}_i = \mathbb{E}\langle x_i^2\rangle E_s/N_0 = k_i\Omega_i E_s/N_0$. By dividing in parts the last two equations yields $X_i^2/\Omega_i = k_i\gamma_i/\overline{\gamma}_i$. Substituting this equation in (10), $F_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2)$ can be obtained. Let $\gamma_{\rm sc}$ denote the instantaneous SNR at the output of the SC receiver, i.e., $\gamma_{\rm sc} = \max(\gamma_1,\gamma_2)$, and $\gamma_{\rm th}$ be a predefined outage threshold. The PDF of $\gamma_{\rm sc}$ can be obtained as $F_{\gamma_{\rm sc}} = F_{\gamma_1,\gamma_2}(\gamma,\gamma)$, while the outage probability at the output of SC can be easily

TABLE I
NUMBER OF TERMS REQUIRED FOR CONVERGENCE OF (11) WITH ACCURCY ± 1‰.

	$\rho_R = 0.1$		$\rho_R = 0.8$	
$\overline{\gamma}(dB)$	$k=1$	$k=4$	$k=1$	$k=4$
-15	3	1	7	2
-10	4	1	11	3
-5	4	2	14	6
0	6	4	18	12
5	8	7	23	20

obtained as $P_{out}(\gamma_{th}) = F_{\gamma_{sc}}(\gamma_{th})$ resulting to

$$F_{X_1,X_2}(x_1,x_2) = \pi^2(1-\rho_R)\sum_{t=0}^{\infty}\rho_R^t\prod_{i=1}^{2}\csc\left[\pi\left(k_i-t+1\right)\right] \times\left\{\frac{\zeta_i^{t+1}}{\Gamma(k_i)}{}_1\tilde{F}_2\left[t+1;t+2-k_i,t+2;\zeta_i\right] -\frac{\zeta^{k_i}}{t!}{}_1\tilde{F}_2\left[k_i;k_i-t,k_i+1;\zeta_i\right]\right\} \tag{11}$$

with $\zeta_i = k_i\gamma_{th}/\left[(1-\rho_R)\overline{\gamma}_i\right]$. For small values of ζ_i, the above equation can be reduced to

$$P_{out}(\gamma_{th}) = \pi^2(1-\rho_R)\sum_{t=0}^{\infty}\rho_R^t\prod_{i=1}^{2}\zeta_i^{t+1}\frac{\csc\left[\pi\left(k_i-t+1\right)\right]}{\Gamma(k_i)t!} \times\left\{\frac{1}{\Gamma(t+2-k_i)(t+1)}-\frac{\zeta_i^{k_i-t-1}}{\Gamma(k_i-t)k_i}\right\} \tag{12}$$

which includes only elementary functions.

IV. NUMERICAL RESULTS

In this section a table checking the infinite series convergence rate and some numerical evaluated results are presented. In Table I the number of terms needed for (11), in order to achieve accuracy better than 1‰ after the truncation of the infinite series, is summarized. Hence, the number of terms is presented versus $\overline{\gamma} = \overline{\gamma}_1 = \overline{\gamma}_2$, and for several values of ρ_R and $k = k_1 = k_2$. As it is depicted in Table I, the required terms increase as $\overline{\gamma}$ and/or k decrease, and/or ρ_R increases. In Fig. 1, a few curves for the outage probability as a function of the first branch normalized outage threshold, $\gamma_{th}/\overline{\gamma}_1$, of a dual branch SC receiver operating over correlated K fading channels, are plotted for non id fading conditions $k_2 = k_1/2$ and $\overline{\gamma}_2 = \overline{\gamma}_1/\sqrt{e}$. In Fig. 1, the OP is plotted for several values of k_i, ρ_R and both id and non id fading conditions. In all cases it is depicted that as $\gamma_{th}/\overline{\gamma}_1$ decreases, the outage performance improves. Moreover, the performance improves as ρ_R decreases and/or k_1 increases. As expected, the best performance is obtained for id fading channels.

V. CONCLUSIONS

A bivariate mixture fading channel model was introduced, namely as bivariate K. Novel analytical expressions for the bivariate K distribution were obtained and used to study the outage performance of dual-branch selection diversity receivers.

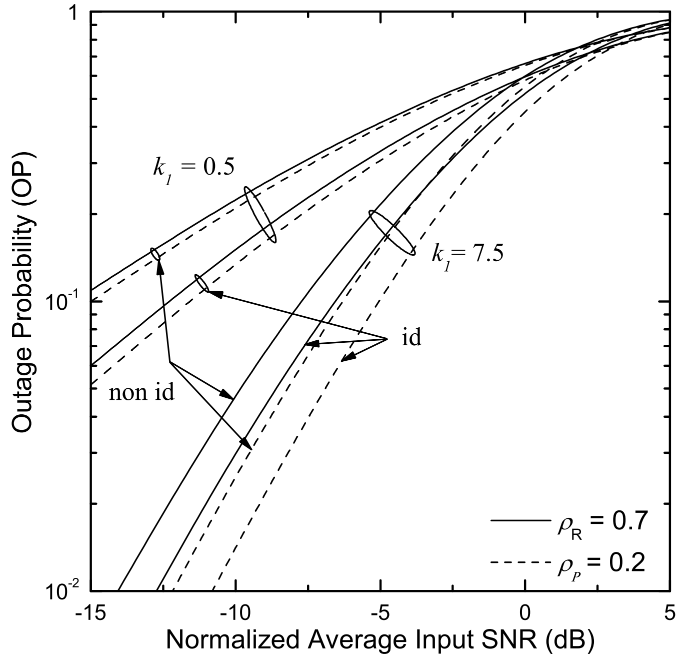

Fig. 1. Outage probability (P_{out}) at the output of a dual-branch SC diversity receiver versus the first branch normalized outage threshold, $\gamma_{th}/\overline{\gamma}_1$, for several values of k_1 and ρ_R.

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