VSP International
Science Publishers
P.O. Box 346, 3700 AH Zeist
The Netherlands

Lecture Series on Computer and Computational Sciences
Volume 1, 2004, pp. 1-3

Information leakage in a quantum computing prototype system: stochastic noise in a microcavity

L.A.A. Nikolopoulos[1] **and A. Maras**

Department of Communication Sciences and Technology,
Faculty of Sciences and Technology,
University of Peloponnese,
GR-221 00 Tripolis, Greece

Received 01 July, 2004; accepted in revised form 01 July 2004

Abstract: One serious problem in the physical implementation of quantum computing is that of maintaining quantum noise. Noise (decoherence) of a quantum system, due to its interaction with the surrounding environment, is the main source of spoiling in the execution of a quantum algorithm. In this work we investigate quantum noise through methods from classical and quantum information theory. Decoherence is seen as the noise effect of a channel operation on a single quantum bit (qubit) and its properties are investigated through the quantum entropy evolution and fidelity transmission. One of the fundamental physical systems that appears promising for quantum computation is that of an atom and electromagnetic (E/M) field in a cavity [1].

Keywords: Quantum information, quantum computation, quantum communication, entanglement, quantum entropy, decoherence, density matrix

PACS: 03.67.Pp, 03.67.Kx, 5.40.-a

1 Introduction

Quantum computation is a rapidly growing field of modern science [2] which has attracted a large number of scientists from areas other than physics, such as mathematics, computing, information and communication theory. Quantum computer is nothing else than a physical system whose evolution can be manipulated in such way so that to perform specific computational tasks. Current technology is faced with the implementation of practical quantum computation algorithms, which when compared with the classical counterpart algorithms (i.e. prime factorization, discrete Fourier transform), achieve an amazing computational performance [3, 4]. One of the main obstacles of quantum computational systems and algorithms is the inevitable imprecision due to interaction of the fundamental quantum system with its environment. Decoherence appears in any quantum computation resulting in a loss of information when a quantum signal is sent in time and/or space through noisy quantum channels. Theoretical and experimental methods have to be developed for the control of such problems which either have pure quantum mechanical origin or they have their analogue in classical information field.

Noise (decoherence) in a quantum physical system

An important topic of research in an open quantum system is the dynamics, induced by the surrounding environment. Noise in such quantum systems, as is the system that we are going to consider, can be

[1]Corresponding author. E-mail: nlambros@uop.gr

described by a master equation for the density operator of the system. Assuming that the total hamiltonian (system + environment) is given by the the hamiltonian $H = H_S + H_E + H_{SE}$, the time evolution of the total density operator is given as the solution of the Liouville equation, where the initial density operator is considered to be in a factorized form as $\rho(0) = \rho_s(0) \otimes \rho_e(0)$. Here H_{SE} denotes the system-environment interaction and ρ_s, ρ_e the density operators of the system and the environment, respectively.
Tracing out the environmental degrees of freedom together with a Markov and a Born approximation [5] results in the following reduced density matrix equation for the system alone:

$$\frac{\partial}{\partial t}\rho_s(t) = -i[H_S, \rho_s] + \sum_{i=a,c}\left[L_i\rho_s L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho_s\}\right], \quad i = a, c. \tag{1}$$

The first term of the right-hand-side describes the coherent evolution of the system (noiseless channel) while the second and third terms ($i = a, c$) describe the noise inserted into the system. The Lindblad operators $L_i = \sqrt{k_i}\sigma_i, i = a, c$ [6] describe the decoherence of the system due to its interaction with the environment. Decoherence time, in the first approximation, is given by the quantity $1/k_i, i = a, c$. Operators σ_a, σ_c represent the destruction operators in the Hilbert space of the system.

Noise in quantum computation

Assume that the initial state $|\psi_s(0)\rangle$ of the system is the input (qubit) of some arbitrary quantum computing operation. The system's evolution, in absence of noise, denote the desired quantum operation (i.e. AND, NOT, XOR, CNOT). The final state of the system $|\psi_s(t_f)\rangle$ represents the value of the qubit after the operation. Noise (decoherence) is now inserted due to the fact that no quantum system is completely isolated from the environment. This affects the final state of the system (final value of the qubit) in a unpredictable way. Let's now consider the case that we are going to examine:

- Evolution of a pure initial state to a final (mixed or not) state.

Assume that the initial state of the system $|\psi_s(0)\rangle = |\psi_a(0)\rangle \otimes |\psi_c(0)\rangle$ is pure and the atom and the cavity field are uncorrelated. As the system evolves according to (Eq. 1), in the presence of the environment, it becomes entangled (quantum correlated) in such way that the state of the system is not possible to be written in a factorized form (uncorrelated case). One interesting question is under what conditions in terms of the input (initial) states of the field and the atom the degree of the entanglement can be maximum.

Noise examined with quantum information concepts

From the quantum information point of view, noise effects may be investigated through the concepts of quantum entropy and the density matrix of the state. More over, fidelity of entanglement as a measure of the pure quantum correlation (non-classical) between the two parts (qubits) of the system is considered. The (Von Neumann) quantum entropy S of the bipartite system is defined as:

$$S(\rho_s) = -Tr(\rho_s \log_2 \rho_s) = -\sum_i r_i log_2 r_i, \tag{2}$$

with r_i being the eigenvalues of the density matrix of the system. The classical counterpart is the Shannon entropy [7] ($S_c = -\sum p_i \log_2 p_i$). It has been shown that in general $S \leq S_c$.

Evolution of the state of the system $\rho_s(t)$ may be modeled as a quantum channel. The initial state of the system $\rho_s(0)$, is the state 'sent' by the source and the final state $\rho_s(t_f)$ is the information 'received' by the receiver. Environmental noise (quantum decoherence) again may affect the final state of the system (receiver readout) in an unpredictable way, thus leading to loss of information.

Here we are going to examine the evolution of the linear entropy (S_l) and the fidelity of entanglement (F_s), defined by:

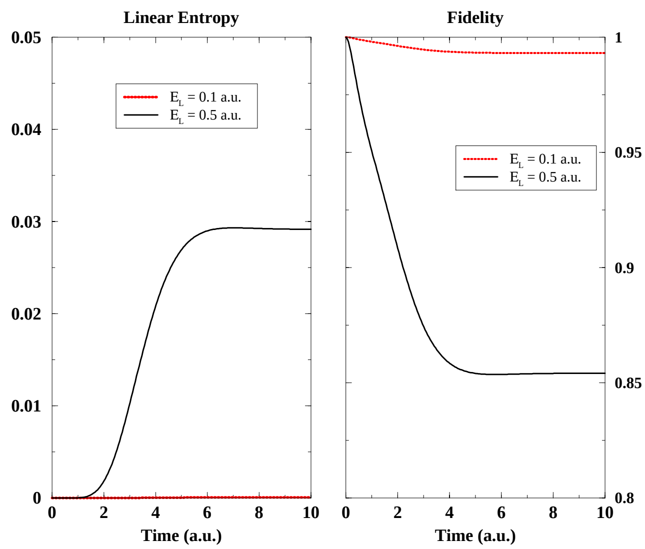

Figure 1: 1 Fidelity transmission and linear entropy for two different constant values of the external E/M field. We have used $\kappa = 2$ and $\gamma = 0.2$ in a.u.

$$S_l = 1 - Tr_s[\rho_s^2(t)] \qquad F_s = \langle\psi_s(t)|\rho_s(0)|\psi_s(t)\rangle = Tr_s[\rho_s(t)\rho_s(0)]. \tag{3}$$

2 Atom and photon in a E/M cavity: quantum computation in a noisy environment

The system is considered to be bipartite, constituted by a typical two-level atomic system (TLS), with hamiltonian $h_a = \omega_0\sigma_z$, inside an E/M cavity modeled by the hamiltonian $h_c = \omega_c a^\dagger a$. Their interaction is given by the operator $h_{ac} = ig(\sigma_+ a + a^\dagger\sigma_-)$. In addition the system is driven by an external E/M field of frequency ω_L with hamiltonian $h_f = \mathcal{E}_L(t)(a^\dagger e^{i\omega_L t} + ae^{-i\omega_L t})$. The atom-cavity field hamiltonian is given by $H = h_a + h_c + h_{ac} + h_l$. In the interaction picture and by making the rotating wave approximation, we end up with the following transformed form for the system hamiltonian:

$$H_S \to H_s = \Delta_a\sigma_+\sigma_- + \Delta_c\alpha^\dagger\alpha + ig(\alpha^\dagger\sigma_- + \sigma_+\alpha) + \mathcal{E}_L(t)(\alpha^\dagger + \alpha), \tag{4}$$

with $\Delta_a = \omega_L - \omega_0$ and $\Delta_c = \omega_L - \omega_c$ the detunings.

Environment, for this bipartite system, is the continuum of E/M field modes surrounding the atom and the cavity walls. Noise is inserted through the purely quantum mechanic phenomenon of spontaneous emission of the excited state of any atomic system. The excited state of the TLS decays into the ground state with a rate γ. Moreover, coupling of the intracavity E/M field with the walls (non-ideal mirrors) of the cavity cause photons to leak out the cavity mirrors with a rate κ. Noise for the atomic system observables modeled with the Pauli destruction spin matrix while noise for the cavity E/M mode, is modeled by the destruction cavity mode operator, namely,

$$L_a = \sqrt{k_a}\ \mathbf{1}(N_c) \otimes \sigma_a(2) = \gamma\,\sigma_-, \qquad L_c = \sqrt{k_c}\ \sigma_c(N_c) \otimes \mathbf{1}(2) = \sqrt{2\kappa}\,\alpha_c.$$

The number N_c is the number of photons for the cavity mode, here kept equal to five $N_c = 5$. The decoherence characteristic times are of the order $\tau_i = 1/k_i, i = a, c$. Let's now assume the initial state of the intracavity field to be the vacuum state $|0_c\rangle$, and the initial state of the atomic system to be in the ground state $|0_a\rangle$. In other words, if we consider that the initial qubit is of the form, $\psi_a(0) = \alpha|0_a\rangle + \beta|1_a\rangle$, then $\alpha = 1, \beta = 0$.

The initial state of the bipartite system (atom+intracavity field) is given by $\psi_s(0) = |\psi_a(0)\rangle \otimes |0_c\rangle = |0_a\, 0_c\rangle$. The initial entropy of the system is $S(0) = 0$. We have complete information for both of the subsystems independently. The two systems are initially uncorrelated.

The system after sufficient time has been evolved in its steady state ψ_{ss} under the influence of the external E/M field of strength $\mathcal{E}_L$ and the noise induced by the environment, characterized by the parameters κ, γ. Evolution is governed by the differential equation for the density matrix elements (Eq. 1). At that time, the two systems are correlated in a non-classical way and their joint (not-separable) state is called the entangled state. Of crucial importance is the entropy that this entangled state has obtained since it represents the amount of information that we can extract from that state. In addition the fidelity of the transmission gives us a quantitative measure of how well the information has been sent from the 'source' to the 'receiver' or equivalently from the quantum computation point of view, the probability that no error has occurred during the execution of the quantum logical algorithm (gate).

In the present case we assume $\kappa = 2,\ \gamma = 0.2$ (characterize the coupling of the noisy environment with the channel). The coupling between the atom and the cavity E/M field has been set to $g = 1$ (characterizes the transmission channel). We also assume that the detunings are zero ($\Delta_c = \Delta_a = 0$). Thus, we have resonant conditions.

For the case of weak external field $\mathcal{E}_L = 0.1$ a.u. we see that the linear entropy S_l remains very close to zero while the fidelity remains very close to one. This suggests that a small amount of information has been lost to the environment while at the same time the system has been entangled. We have almost complete information both for the joint system and for the subsystems independently. The two systems are almost uncorrelated. When we increase the external field to the value of $\mathcal{E}_L = 0.5$ a.u. we see that entropy is increased in its steady value but still stays very small ($S_l \to 0.029$) while fidelity decreases more rapidly with the field strength ($F_s \to 0.854$). This is consequence of the fact that while the information loss for the joint system is small the available information for the the two subsystems separately has been decreased. The degree of the entanglement (degree of quantum correlation) between the two subsystems has been increased. Results for different initial states and other envriromental parameters have also been obtained.

References

[1] B. B. Blinov, D. L. Moehring, L.-M. Duan, and C. Monroe, Observation of entanglement between a single trapped atom and a single photon. *Nature*, 428:153–157, 2004.

[2] C. Monroe, Quantum information processing with atoms and photons. *Nature*,416:238–246, 2002.

[3] P. Shor. Scheme for reducing decoherence in quantum computer memory. *Phys. Rev. A*, 52:R2493, 1995.

[4] Lov K. Grover. Quantum mechanics helps in searching for a needle in a haystack. *Phys. Rev. Lett.*, 78:4709–4712, 1997.

[5] N.G. Van Kampen. *Stochastic Processes in Physics and Chemistry*. Elsevier Science B.V. Amsterdam, 1992.

[6] G. Lindblad. On the generators of quantum dynamical semigroups. *Commun. Math. Phys.*, 48:119–130, 1976.

[7] C.E. Shannon. A mathematical theory of communication. *Bell, Syst. Tech. J.*, 27:379–423,623–656, 1948.