Λύσεις σειράς ασκήσεων 3

1. Να βρεθούν τα παρακάτω ολοκληρώματα:

β) $\int x^6 e^{-2x} dx$

Εφαρμόζουμε την πινακοειδή ολοκλήρωση και έχουμε στην πρώτη στήλη την x^6 και τις παραγώγους της και στη δεύτερη στήλη την e^{-2x} και τα ολοκληρώματά της: Επομένως:

x^6	πρόσημο	e^{-2x}
x^6	$+$	e^{-2x}
$6x^5$	$-$	$-e^{-2x}/2$
$30x^4$	$+$	$e^{-2x}/4$
$120x^3$	$-$	$-e^{-2x}/8$
$360x^2$	$+$	$e^{-2x}/16$
$720x$	$-$	$-e^{-2x}/32$
720	$+$	$e^{-2x}/64$
0	$-$	$-e^{-2x}/128$

$$\int x^6 e^{-2x} dx = -\left(\frac{x^6}{2} + \frac{3x^5}{2} + \frac{15x^4}{4} + \frac{15x^3}{2} + \frac{45x^2}{4} + \frac{45x}{4} + \frac{45}{8}\right) e^{-2x}$$

ϑ) $\int_0^{\pi/4} x \tan^2(x) dx$

$$\int_0^{\pi/4} x \tan^2(x) dx = \int_0^{\pi/4} x \frac{\sin^2(x)}{\cos^2(x)} dx = \int_0^{\pi/4} x \frac{1-\cos^2(x)}{\cos^2(x)} dx = \int_0^{\pi/4} \frac{x}{\cos^2(x)} dx - \int_0^{\pi/4} x dx =$$

$$= \int_0^{\pi/4} x \left(\tan(x)\right)' dx - \left[\frac{x^2}{2}\right]_0^{\pi/4} = [x \tan(x)]_0^{\pi/4} - \int_0^{\pi/4} \tan(x) dx - \frac{\pi^2}{32} =$$

$$= \frac{\pi}{4} - \int_0^{\pi/4} -\frac{(cos(x))'}{\cos(x)} dx - \frac{\pi^2}{32} = \frac{\pi}{4} - \frac{\pi^2}{32} + [\ln|\cos(x)|]_0^{\pi/4} = \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2}\ln 2.$$

ι) $\int 4x \sec^2(2x) dx$

$$\int 4x \sec^2(2x) dx = \int 2x (\tan(2x))' dx = 2x \tan(2x) - 2\int \tan(2x) dx = 2x\tan(2x) - \ln|\cos(2x)|.$$

κ) $\int e^{\sqrt{4x+9}}dx$

Θέτουμε $y=\sqrt{4x+9}$. Άρα: $dx=(y/2)dy$ και

$$\int e^{\sqrt{4x+9}}dx=\frac{1}{2}\int ye^{y}dy=\frac{1}{2}\int y(e^{y})'dy=\frac{1}{2}ye^{y}-\frac{1}{2}\int e^{y}dy=\frac{1}{2}ye^{y}-\frac{1}{2}e^{y}=\frac{1}{2}\left(\sqrt{4x+9}-1\right)e^{\sqrt{4x+9}}+C$$

2. Να βρεθούν τα παρακάτω ολοκληρώματα:

ε) $\int_0^1 \frac{dx}{(x+1)(x^2+1)}$

$$\frac{1}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}\Rightarrow 1=(x+1)(Bx+C)+A(x^2+1)\Rightarrow$$

$$(A+B)x^2+(B+C)x+(A+C)=1\Rightarrow A+B=B+C=0,\ A+C=1\Rightarrow A=C=1/2,\ B=-1/2\Rightarrow$$

$$\int_0^1\frac{dx}{(x+1)(x^2+1)}=\int_0^1\left(\frac{1}{2(x+1)}-\frac{x-1}{2(x^2+1)}\right)dx=$$

$$=\int_0^1\frac{dx}{2(x+1)}-\int_0^1\frac{x}{2(x^2+1)}dx+\int_0^1\frac{1}{2(x^2+1)}dx=\frac{1}{2}\left[\ln|x+1|\right]_0^1-\int_0^1\frac{(x^2+1)'}{4(x^2+1)}dx+\frac{1}{2}\left[\arctan(x)\right]_0^1=$$

$$\frac{1}{2}\ln 2-\frac{1}{4}\left[\ln|x^2+1|\right]_0^1+\frac{\pi}{8}=\frac{1}{4}\ln 2-\frac{\pi}{8}.$$

3. Να βρεθούν τα παρακάτω ολοκληρώματα:

α) $\int_0^{2\pi}\sqrt{\frac{1-\cos(2x)}{2}}dx$

Από την τριγωνομετρική ταυτότητα έχουμε:

$$\int_0^{2\pi}\sqrt{\frac{1-\cos(2x)}{2}}dx=\int_0^{2\pi}\sqrt{\sin^2(x)}dx=\int_0^{2\pi}|\sin(x)|dx=\int_0^{\pi}\sin(x)dx-\int_{\pi}^{2\pi}\sin(x)dx=$$

$$[\cos(x)]_0^{\pi}-[\cos(x)]_{\pi}^{2\pi}=4.$$

ε) $\int_0^{\pi/2}\sin^5(x)dx$

$$\int_0^{\pi/2}\sin^5(x)dx=\int_0^{\pi/2}\sin^4(x)\sin(x)dx=\int_0^{\pi/2}(1-\cos^2(x))^2\sin(x)dx=$$

$$= -\int_0^{\pi/2} (1+\cos^4(x) - 2\cos^2(x))(\cos(x))' dx = -\left[\cos(x) + \frac{\cos^5(x)}{5} - \frac{2\cos^3(x)}{3}\right]_0^{\pi/2} = \frac{8}{15}.$$

η) $\int \cot^5(x) dx$

Έχουμε: $\cot^2(x) = \frac{\cos^2(x)}{\sin^2(x)} = \frac{1-\sin^2(x)}{\sin^2(x)} = \csc^2(x) - 1$. Άρα:

$$\int \cot^5(x) dx = \int \cot^2(x)\cot^3(x) dx = \int (\csc^2(x) - 1)\cot^3(x) dx =$$

$$= -\int (\cot(x))' \cot^3(x) dx - \int \cot^3(x) dx = -\frac{1}{4}\cot^4(x) - \int \cot^2(x)\cot(x) dx =$$

$$= -\frac{1}{4}\cot^4(x) - \int (\csc^2(x) - 1)\cot(x) dx = -\frac{1}{4}\cot^4(x) + \int (\cot(x))' \cot(x) dx + \int \cot(x) dx =$$

$$= -\frac{1}{4}\cot^4(x) + \frac{1}{2}\cot^2(x) + \ln|\sin(x)| + C.$$

ϑ) $\int \csc(x) dx$

$$\int \csc(x) dx = \int \csc(x) \cdot \frac{\csc(x) + \cot(x)}{\csc(x) + \cot(x)} dx = \int \frac{\csc^2(x) + \csc(x)\cot(x)}{\csc(x) + \cot(x)} dx =$$

$$= -\int \frac{(\csc(x) + \cot(x))'}{\csc(x) + \cot(x)} dx = -\ln|\csc(x) + \cot(x)| + C.$$

ι) $\int (\sec(x) + \cot(x))^2 dx$

$$\int (\sec(x) + \cot(x))^2 dx = \int (\sec^2(x) + \cot^2(x) + 2\sec(x)\cot(x)) dx =$$

$$= \int \sec^2(x) dx + \int \cot^2(x) dx + 2\int \frac{\cos(x)}{\sin^2(x)} dx = \tan(x) + \int (\csc^2(x) - 1) dx + 2\int \frac{(\sin(x))'}{\sin^2(x)} dx =$$

$$= \tan(x) - \cot(x) + x - \frac{2}{\sin(x)} + C.$$

4. Να βρεθούν τα παρακάτω ολοκληρώματα:

γ) $\int \frac{dx}{(1+x^2)\tan^{-1}(x)}$

$$\int \frac{dx}{(1+x^2)\tan^{-1}(x)} = \int (\arctan(x))' \arctan(x)dx = (1/2)\arctan^2(x) + C$$

δ) $\int \frac{dx}{x^2-2x+5}$

$$\int \frac{dx}{x^2-2x+5} = \int \frac{dx}{x^2-2x+1+4} = \int \frac{dx}{(x-1)^2+2^2} = (1/2)\arctan((x-1)/2) + C.$$

ε) $\int \frac{x^3dx}{\sqrt{x^2+4}}$

Θέτουμε: $x = 2\tan(\theta) \Rightarrow dx = 2\sec^2(\theta)d\theta$ και έχουμε:

$$\int \frac{x^3dx}{\sqrt{x^2+4}} = \int \frac{8\tan^3(\theta)2\sec^2(\theta)}{\sqrt{4\tan^2(\theta)+4}}d\theta = 8\int \frac{\tan^3(\theta)2\sec^2(\theta)}{\sqrt{\sec^2(\theta)}}d\theta = 8\int \tan^3(\theta)\sec(\theta)d\theta =$$

$$= 8\int \frac{\sin^3(\theta)}{\cos^4(\theta)}d\theta = 8\int \frac{\sin^2(\theta)\sin(\theta)}{\cos^4(\theta)}d\theta = -8\int \frac{(1-\cos^2(\theta))(\cos(\theta))'}{\cos^4(\theta)}d\theta =$$

$$= -8\int \frac{(\cos(\theta))'}{\cos^4(\theta)}d\theta + 8\int \frac{(\cos(\theta))'}{\cos^2(\theta)}d\theta = \frac{8}{3\cos^3(\theta)} - \frac{8}{\cos(\theta)} + C.$$

Επίσης: $x = 2\tan(\theta) \Rightarrow x^2 = 4\tan^2(\theta) = 4\sec^2(\theta) - 4 = 4/\cos^2(\theta) - 4 \Rightarrow \cos(\theta) = 2/\sqrt{x^2+4}$. Αντικαθιστώντας στην έκφραση για το ολοκλήρωμα:

$$\int \frac{x^3dx}{\sqrt{x^2+4}} = \frac{8(x^2+4)^{3/2}}{3\cdot 8} - \frac{8\sqrt{x^2+4}}{2} + C = \frac{1}{3}(x^2-8)\sqrt{x^2+4} + C.$$

ζ) $\int \frac{x^2dx}{(x^2-1)^{5/2}}$

Θέτουμε: $x = \sec(\theta) \Rightarrow dx = \sec(\theta)\tan(\theta)d\theta$ και έχουμε:

$$\int \frac{x^2dx}{(x^2-1)^{5/2}} = \int \frac{\sec^3(\theta)\tan(\theta)}{(\sec^2(\theta)-1)^{5/2}}d\theta = \int \frac{\sec^3(\theta)2\tan(\theta)}{(\tan^2(\theta))^{5/2}}d\theta = \int \frac{\sec^3(\theta)}{\tan^4(\theta)}d\theta =$$

$$= \int \frac{\cos(\theta)}{\sin^4(\theta)}d\theta = 8\int \frac{(\sin(\theta))'}{\sin^4(\theta)}d\theta = -\frac{1}{3\sin^3(\theta)} + C.$$

Επίσης: $x = \sec(\theta) \Rightarrow x^2 = 1/\cos^2(\theta) \Rightarrow \sin(\theta) = \sqrt{1-\cos^2(\theta)} = \sqrt{x^2-1}/x$. Αντικαθιστώντας στην έκφραση για το ολοκλήρωμα:

$$\int \frac{x^2 dx}{(x^2-1)^{5/2}} = -\frac{1}{3\sin^3(\theta)} + C = -\frac{x^3}{3(x^2-1)^{3/2}} + C.$$

η) $\int \frac{\sqrt{9-x^2}}{x^2} dx$

Θέτουμε: $x = 3\sin(\theta) \Rightarrow dx = 3\cos(\theta)d\theta$ και έχουμε:

$$\int \frac{\sqrt{9-x^2}}{x^2} dx = \int \frac{3\cos(\theta)\sqrt{9-9\sin^2(\theta)}}{\sin^2(\theta)} d\theta = 9\int \frac{\cos^2(\theta)}{\sin^2(\theta)} d\theta = 9\int \cot^2(\theta) d\theta =$$

$$= 9\int (1+\csc^2(\theta))d\theta = 9\theta - 9\cot(\theta) + C.$$

Επίσης: $x = 3\sin(\theta) \Rightarrow x^2 = 9\sin^2(\theta) \Rightarrow \cos(\theta) = \sqrt{9-x^2}/3 \Rightarrow \cot(\theta) = \sqrt{9-x^2}/x$. Αντικαθιστώντας στην έκφραση για το ολοκλήρωμα:

$$\int \frac{\sqrt{9-x^2}}{x^2} dx = 9\arcsin(x/3) - 9\sqrt{9-x^2}/x + C.$$

5. Να υπολογιστούν οι παράγωγοι των κάτωθι συναρτήσεων:

γ) $(1+x^2)\coth^{-1}(x^2)$

$$\left((1+x^2)\coth^{-1}(x^2)\right)' = (1+x^2)'\coth^{-1}(x^2) + (1+x^2)(\coth^{-1}(x^2))' =$$

$$= 2x\coth^{-1}(x^2) + \frac{(1+x^2)(x^2)'}{1-(x^2)^2} = 2x\coth^{-1}(x^2) + \frac{2x(1+x^2)}{(1-x^2)(1+x^2)} = 2x\coth^{-1}(x^2) + \frac{2x}{1-x^2}.$$

δ) $\sinh^{-1}(x^{3/2})$

$$\left(\sinh^{-1}(x^{3/2})\right)' = \frac{(x^{3/2})'}{\sqrt{1+(x^{3/2})^2}} = \frac{3\sqrt{x}}{2\sqrt{1+x^3}}.$$

8. Να βρεθούν τα παρακάτω ολοκληρώματα:

α) $\int_2^\infty \frac{1}{x^2-1}dx$

Ορίζουμε το:

$$I_b = \int_2^b \frac{1}{x^2-1}dx = \int_2^b \frac{1}{(x-1)(x+1)}dx = \int_2^b \left(\frac{1}{2(x-1)} - \frac{1}{2(x+1)} \right) dx =$$

$$= \frac{1}{2}\left[\ln(x-1)\right]_2^b - \frac{1}{2}\left[\ln(x+1)\right]_2^b = \frac{1}{2}\ln(b-1) - \frac{1}{2}\ln(b+1) + \frac{1}{2}\ln 3 = \frac{1}{2}\ln\left(\frac{b-1}{b+1}\right) + \frac{1}{2}\ln 3.$$

$$I = \lim_{b\to\infty} I_b = \lim_{b\to\infty} \frac{1}{2}\ln\left(\frac{b-1}{b+1}\right) + \frac{1}{2}\ln 3 = \lim_{b\to\infty} \frac{1}{2}\ln\left(\frac{1}{1}\right) + \frac{1}{2}\ln 3 = \frac{1}{2}\ln 3.$$