5. MATRICES AND DETERMINANTS

Objectives

To be able to perform basic matrix operations.

To recognise the importance of the square matrix and to be able to find its detrminant and inverse.

To be able to use matrix methods to solve simultaneous equations and to find eigen-values and eigen-vectors.

5.1 Algebra of Matrices.

A **matrix** is a set of mn quantities arranged in m rows and n columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{pmatrix}$$

The individual elements are referred to as a_{ij} where i represents the row and j represents the column. The matrix is said to have **order** mn.

Addition and Subtraction

In order that matrices can be added and subtracted they must be of the same order, that is, the same shape and size. The entries are then added or subtracted one to one.

eg $A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 7 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 2 & -1 \\ 0 & 3 & 2 \end{pmatrix}$

$A + B = \begin{pmatrix} 7 & 5 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ $A - B = \begin{pmatrix} -3 & 1 & 6 \\ 1 & -3 & 5 \end{pmatrix}$

Multiplication by a scalar

We can multiply and divide matrices by a *scalar* which is a quantity k, just as we do with ordinary numbers. Each entry in the matrix is multiplied by the scalar.

eg $kA = k \begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 2k & 3k & 5k \\ k & 0 & 7k \end{pmatrix}$

If k = 3 then $3A = \begin{pmatrix} 6 & 9 & 15 \\ 3 & 0 & 21 \end{pmatrix}$ if k = 1/2 then $A/2 = \begin{pmatrix} 1 & 3/2 & 5/2 \\ 1/2 & 0 & 7/2 \end{pmatrix}$

Matrix Multiplication

Two matrices A and B can be multiplied together to form a new matrix AB only if

Number of columns in A = Number of rows in B

So that if the matrix A has order **(m x p)**, the matrix B has order **(p x n)** then the matrix AB has order **(m x n)**. To carry out matrix multiplication you must follow the method below, it is based on an **across** and **down** pattern.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{pmatrix}$$

eg $A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{pmatrix}$

$$AB = \begin{pmatrix} 3.1 + 1.3 + 2.2 & 3.2 + 1.1 + 2.3 \\ 2.1 + 1.3 + 3.2 & 2.1 + 1.1 + 3.3 \end{pmatrix} = \begin{pmatrix} 10 & 13 \\ 11 & 14 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1.3 + 2.2 & 1.1 + 2.1 & 1.2 + 2.3 \\ 3.3 + 1.2 & 3.1 + 1.1 & 3.2 + 1.3 \\ 2.3 + 3.2 & 2.1 + 3.1 & 2.2 + 3.3 \end{pmatrix} = \begin{pmatrix} 7 & 3 & 8 \\ 11 & 4 & 9 \\ 12 & 5 & 13 \end{pmatrix}$$

Using the matrices below, evaluate AB, BA, AC, CA, BC, CB if possible.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 7 & 9 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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[Solution: only AB, CA and BC work]

5.2 Matrix Operations

The Square Matrix

A square matrix has the same number of rows and columns so it has order (n x n). The diagonal containing entries a_{11}, a_{22}, ..., a_{nn} is called the *leading diagonal* and its sum is called the *trace* of the matrix.

eg $A = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 9 & 7 \\ -2 & 6 & 3 \end{pmatrix}$ $\quad Tr(A) = 1 + 9 + 3 = 13$

The Diagonal Matrix

This is a square matrix with all the entries except those on the leading diagonal equal to zero.

eg $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is a diagonal matrix

The Identity Matrix

This is a diagonal matrix with all the entries on the leading diagonal equal to one. The abbreviation for this matrix is I_n or usually just I.

eg $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the 3 x 3 identity matrix.

This matrix is important because AI = IA = A. Multiplying any matrix A by I leaves A unchanged.

Determinants

The determinant may only be defined for a square matrix and it is denoted | A |.

eg $|A| = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = 2(-2) - 1.3 = -7$

For the general matrix A, this is

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ae - bd$$

To find the determinant of a 3 x 3 matrix, consider the general 3 x 3 matrix below and the calculation above carried out three times.

eg $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Notice that we take each of the entries in the top row and multiply them by the determinant of the entries *not* in that row or column. This is called the **cofactor** of that entry.

ie $\begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei - fh$ is the cofactor of **a**

The three determinants are not just added to find the total determinant, the sign of the cofactor is chosen using a 'chequer-board' as below.

For a 3 x 3 matrix

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

For a 4 x 4 matrix

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

So the determinant of a 4 x 4 matrix B is given by the formula below.

$$|B| = \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

$$= af\begin{vmatrix} k & l \\ o & p \end{vmatrix} - ag\begin{vmatrix} j & l \\ n & p \end{vmatrix} + ah\begin{vmatrix} j & k \\ n & o \end{vmatrix} + be\begin{vmatrix} k & l \\ o & p \end{vmatrix} + bg\begin{vmatrix} i & l \\ m & p \end{vmatrix} - bh\begin{vmatrix} i & k \\ m & o \end{vmatrix}$$
$$- ce\begin{vmatrix} j & l \\ n & p \end{vmatrix} + cf\begin{vmatrix} i & l \\ m & p \end{vmatrix} + ch\begin{vmatrix} i & j \\ m & n \end{vmatrix} + de\begin{vmatrix} j & k \\ n & o \end{vmatrix} - df\begin{vmatrix} i & k \\ m & o \end{vmatrix} + dg\begin{vmatrix} i & j \\ m & n \end{vmatrix}$$

eg $|A| = \begin{vmatrix} 1 & 3 & 5 & -2 \\ 0 & 6 & 1 & 2 \\ -3 & 0 & 5 & -3 \\ 2 & -1 & 4 & 0 \end{vmatrix}$

$= 6\begin{vmatrix} 5 & -3 \\ 4 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & -3 \\ 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 0 & 5 \\ -1 & 4 \end{vmatrix} + 3\begin{vmatrix} -3 & -3 \\ 2 & 0 \end{vmatrix} - 3.2\begin{vmatrix} -3 & 5 \\ 2 & 4 \end{vmatrix}$

$+ 5.6\begin{vmatrix} -3 & -3 \\ 2 & 0 \end{vmatrix} + 5.2\begin{vmatrix} -3 & 0 \\ 2 & -1 \end{vmatrix} + 2.6\begin{vmatrix} -3 & 5 \\ 2 & 4 \end{vmatrix} - 2\begin{vmatrix} -3 & 0 \\ 2 & -1 \end{vmatrix}$

$= 6.12 + 3 + 2.5 + 3.6 - 6(-22) + 30.6 + 10.3 + 12(-22) - 2.3$

$= 165$

How many 2 x 2 determinants do you need to work out to find the determinant of a 5 x 5 matrix?

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[Solution: 5 x 4 x 3 = 60]

Transpose

A transposed matrix is one which has its rows and columns interchanged.

It is denoted A^{I} or sometimes A^T.

ie $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$ $\quad A^{I} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix}$

Inverse

There are various ways of finding the inverse of a matrix A. The inverse A^{-1} is such that $AA^{-1} = I$.

The first method described here is the method of **row operations**. We append the identity matrix to the matrix A and carry out the same algebraic manipulations on both A and I. By making A become the identity, I becomes the inverse of A.

eg $A = \begin{pmatrix} 2 & 1 & 1/2 \\ 1 & 0 & 5 \\ 1/2 & 1 & 1 \end{pmatrix}$ $\quad A\,|\,I = \left(\begin{array}{ccc|ccc} \mathbf{2} & 1 & 1/2 & 1 & 0 & 0 \\ 1 & 0 & 5 & 0 & 1 & 0 \\ 1/2 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$ x1/2

Step 1: make a_{11} equal to one by multiplying the whole row 1 by 1/2 and add multiples of this row to the other rows to make the other entries in the first column equal to zero. ie. add (-1)(row 1 values) to the corresponding entries in row 2 etc.

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 1/4 & 1/2 & 0 & 0 \\ 1 & 0 & 5 & 0 & 1 & 0 \\ 1/2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \leftarrow -1 \\ \leftarrow -1/2 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 1/2 & 1/4 & 1/2 & 0 & 0 \\ 0 & \mathbf{-1/2} & 19/4 & -1/2 & 1 & 0 \\ 0 & 3/4 & 7/8 & -1/4 & 0 & 1 \end{array}\right) \text{ x-2}$$

Step 2: make a_{22} equal one and use row operations to make the other entries in column two equal zero.

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 1/4 & 1/2 & 0 & 0 \\ 0 & 1 & -19/2 & 1 & -2 & 0 \\ 0 & 3/4 & 7/8 & -1/4 & 0 & 1 \end{array}\right) \begin{array}{l} \leftarrow -1/2 \\ \\ \leftarrow -3/4 \end{array} \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & 1 & 0 \\ 0 & 1 & -19/2 & 1 & -2 & 0 \\ 0 & 0 & \mathbf{8} & -1 & 3/2 & 1 \end{array}\right) \text{ x1/8}$$

Step 3: make a_{33} equal one and the other entries in the third column equal zero as before.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & 1 & 0 \\ 0 & 1 & -19/2 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1/8 & 3/16 & 1/8 \end{array}\right) \begin{array}{l} \leftarrow -5 \\ \leftarrow 19/2 \\ \\ \end{array} \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/8 & 1/16 & -5/8 \\ 0 & 1 & 0 & -3/16 & -7/32 & 19/6 \\ 0 & 0 & 1 & -1/8 & 3/16 & 1/8 \end{array}\right)$$

Now the first matrix is the identity matrix so the second matrix is the inverse of the original matrix.

Check $AA^{-1} = I$

$$\begin{pmatrix} 2 & 1 & 1/2 \\ 1 & 0 & 5 \\ 1/2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 5/8 & 1/16 & -5/8 \\ -3/16 & -7/32 & 19/16 \\ -1/8 & 3/16 & 1/8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second method uses the **adjoint matrix** below.

$$\text{adj } A = B' = \begin{pmatrix} A_{11} & A_{21} & \cdots\cdots\cdots & A_{n1} \\ A_{12} & A_{22} & \cdots\cdots\cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots\cdots\cdots & A_{nn} \end{pmatrix}$$

A_{ij} are the cofactors of the matrix A and the signs of the cofactors are found using the chequer-board as before.

eg $A = \begin{pmatrix} 2 & 1 & 1/2 \\ 1 & 0 & 5 \\ 1/2 & 1 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} -5 & 3/2 & 1 \\ -1/2 & 7/4 & -3/2 \\ 5 & -19/2 & -1 \end{pmatrix}$

$A_{11} = 0 - 5 = -5$ $\quad A_{12} = -(1 - 5/2) = 3/2$ $\quad A_{13} = 1$

$A_{21} = -(1 - 1/2) = -1/2$ $\quad A_{22} = 2 - 1/4 = 7/4$ $\quad A_{23} = -(2 - 1/2) = -3/2$

$A_{31} = 5 - 0 = 5$ $\quad A_{32} = -(10 - 1/2) = -19/2$ $\quad A_{33} = 0 - 1 = -1$

So $B' = \text{adj } A = \begin{pmatrix} -5 & -1/2 & 1 \\ 3/2 & 7/4 & -19/2 \\ 1 & -3/2 & -1 \end{pmatrix}$

The formula for the inverse is $A^{-1} = \dfrac{\text{adj } A}{|A|}$

$|A| = 2(-5) - (1 - 5/2) + 1/2(1 - 0) = -8$

So $A^{-1} = \dfrac{-1}{8}\begin{pmatrix} -5 & -1/2 & 1 \\ 3/2 & 7/4 & -19/2 \\ 1 & -3/2 & -1 \end{pmatrix} = \begin{pmatrix} 5/8 & 1/16 & -5/8 \\ -3/16 & -7/3 & 19/16 \\ -1/8 & 3/16 & 1/8 \end{pmatrix}$

5.3 Uses of Matrices

Solution of Simultaneous Linear Equations.

This is a simple way of solving simultaneous equations when there are more than two equations in the system.

eg
$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 9z = 36$$

This system can be re-written in matrix form as

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 36 \end{pmatrix}$$

Try multiplying the two matrices, the result will be the system above.

Now, $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 6 \\ 14 \\ 36 \end{pmatrix}$

So, $A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{2} \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{pmatrix}$

Then, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{pmatrix} \begin{pmatrix} 6 \\ 14 \\ 36 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$x = 1$, $y = 2$, $z = 3$ are the solutions.

If the matrix A has $| A | = 0$ then we are unable to solve the system and we conclude that the system is *inconsistent*. Such a matrix A is called **singular**. Conversely, a **non-singular** matrix has $| A | \neq 0$. This method can only be used when there are the same number of equations as there are variables, since we need a square matrix A.

Eigen-values and eigen-vectors

Consider the equation $(A - \lambda I)X = 0$ where A is an n x n matrix, X is a *column vector* of n rows, I is the identity matrix and λ is a parameter.

Solutions exist for $|A - \lambda I| = 0$. This is called the **characteristic equation** of the matrix A and the roots $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are the **eigen-values** of A.

eg Find the eigen-values of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

Then, $(1-\lambda)\,[\,(2-\lambda)(3-\lambda) - 0\,] \;\; -2+\lambda = 0$

$6 - 5\lambda + \lambda^2 - 6\lambda + 5\lambda^2 - \lambda^3 - 2 + \lambda = 0$ This is the characteristic equation

$-\lambda^3 + 6\,^2 - 10\lambda + 4 = 0$

$(\lambda - 2)(-\lambda^2 + 4\lambda - 2) = 0$

$\lambda = 2,\; 2 \pm \sqrt{2}$

To each eigen-value λ, there is a solution X of the system of equations $(A - \lambda I)X = 0$, where X is a coumn vector of order n. These individual solutions are called **eigen-vectors**.

<u>For $\lambda = 2$</u> $(A - \lambda I)X = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

So we have $-x + z = 0 \rightarrow x = z$

$x + y + z = 0$

Let x = 1 then y = 0, z = -1.

So the eigen-vector $X_1 = (1 \;\; 0 \;\; -1)'$ satisfies the equation. Note that you can choose any value for one of the variables but you must make sure that the other variables are consistent with it.

For $\lambda = 2+\sqrt{2}$
$$\begin{pmatrix} -1-\sqrt{2} & 0 & 1 \\ 0 & -\sqrt{2} & 0 \\ 1 & 0 & 1-\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(-1-\sqrt{2})x + z = 0 \quad \rightarrow \quad x = z / (1 + \sqrt{2})$$
$$(-\sqrt{2})y = 0 \quad \rightarrow \quad y = 0$$
$$x + (1 - \sqrt{2})z = 0 \quad \rightarrow \quad x = z(1 + \sqrt{2})$$

Let $x = 1$, then $z = 1+\sqrt{2}$ so $X_2 = (1 \;\; 0 \;\; 1+\sqrt{2})'$

For $\lambda = 2\sqrt{2}$
$$\begin{pmatrix} -1+\sqrt{2} & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1+\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(-1+\sqrt{2})x + z = 0 \quad \rightarrow \quad x = z / (1 - \sqrt{2})$$
$$(\sqrt{2})y = 0 \quad \rightarrow \quad y = 0$$
$$x + (1+ \sqrt{2})z = 0 \quad \rightarrow \quad x = z(-1 - \sqrt{2})$$

Let $x = 1$, then $z = 1-\sqrt{2}$ so $X_3 = (1 \;\; 0 \;\; 1-\sqrt{2})'$

So $X = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix}$ is the solution matrix. ie. $x = (X_1, X_2, X_3)$

The three eigen-vectors X_1, X_2 and X_3 which correspond to the eigen-values λ_1, λ_2, and λ_3 satisfy the equation $(A - \lambda I)X = 0$.

Summary

Matrices are in fact a sequence of column vectors, this is clear from the example on eigen-vectors, so they are ideally suited to problems involving systems of equations. By using matrices, each operation need only be performed once on the whole matrix instead of on each equation.

Matrix algebra is simple, even matrix mulitiplication, as long as a few simple rules are remembered. Two matrices of the same size and shape can be added or subtracted; two matrices can be multiplied together *only* if the number of columns in the first matrix equals the number of rows in the second.

Finding the determinant and the inverse of a matrix is an important technique which is used a lot in more advanced mathematics.

Activities

1. If $A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 5 & 7 \\ 1 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 1 & 0 \\ 6 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ find A+B, A-B, B-A, AB and BA.

2. If $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 3 & -1 \end{pmatrix}$ find A^2.

3. Evaluate the following.

$$|A| = \begin{vmatrix} 0 & 2 & -1 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{vmatrix} \qquad |B| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$|C| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 2 & 1 \\ 1 & 1 & -1 \end{vmatrix} \qquad |D| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

Hence find the determinant below.

$$\begin{vmatrix} -1 & 2 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ 2 & 2 & -1 & 1 \\ 1 & -1 & 2 & 1 \end{vmatrix}$$

4. The 2 x 2 matrices a, b, c and I are defined below.

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If $i^2 = -1$, show that

(i) $a^2 = b^2 = c^2 = I$

(ii) $ab = -ba = ic$
$bc = -cb = ia$
$ca = -ac = ib$

5. Find the inverse of A and B below and check that they are correct.

$$A = \begin{pmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6. Solve the simultaneous equations using matrix methods.

(a) $4x - 3y + z = 11$
$2x + y - 4z = -1$
$x + 2y - 2z = 1$

(b) $x + 5y + 3z = 1$
$5x + y - z = 2$
$x + 2y + z = 3$

7. If $X = \begin{pmatrix} -1/2 & -3/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Show that $P^{-1}XP$ is a diagonal matrix and that X satisfies the characteristic equation

$$X^3 - 2X^2 - X + 2I = 0$$

8. Find the eigen-values of the matrix $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$

9. Find the characteristic equation of the matrices below.

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

10. Find eigen-values and eigen-vectors for the following matrices.

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$$

[Solutions:

3 -9, -10, -3, -5, 34.

6 (a) $x = 3, y = 1, z = 2$; (b) inconsistent.

8 4, -1.

9 All have $-\lambda^3 + 6\lambda^2 + 3\lambda - 18$.

10 (a) $\lambda_1 = 1$, $X_1 = (1\ \ 0\ \ 0)'$; $\lambda_2 = 2$, $X_2 = 1/\sqrt{17}(4\ \ 1\ \ 0)'$; $\lambda_3 = 3$, $X_3 = 1/\sqrt{989}(29\ \ 12\ \ 2)'$; (b) $\lambda_1 = 2$, $X_1 = (1\ \ 0\ \ 0)'$, $\lambda_2 = \lambda_3 = 1$ $X_2 = X_3 = 1/\sqrt{10}(-3\ \ 1\ \ 0)'$; (c) $\lambda_1 = 0$ $X_1 = 1/\sqrt{2}(1\ \ -1\ \ 0)'$; $\lambda_2 = 4$, $X_2 = 1/\sqrt{2}(1\ \ 1\ \ 0)'$, $\lambda_3 = 1$ $X_3 = (0\ \ 0\ \ 1)'$, (d) $\lambda_1 = 0$ $X_1 = 1/\sqrt{3}(-1-i\ \ 1)'$, $\lambda_2 = 3$ $X_2 = 1/\sqrt{6}(1+i\ \ 2)'$.]