8. INTEGRATION

Objectives

To know that integration is the reverse of differentiation an to be able to perform basic integration.

To be able to evaluate the area under a curve using the definite integral.

To understand various techniques of integration and to be able to carry them out.

8.1 Introduction

The reverse of differentiation is called **integration**.

ie $f(x) = x^3 + 1$ $f^{\prime}(x) = 3x^2$

So $x^3 + 1$ is the *integral* of $3x^2$.

Written properly, this is $\int 3x^2\,dx = x^3 + c$

$\int$ means 'the integral of'
dx means 'with respect to x'

Everything between the two signs is integrated. Note the inclusion of the letter c in the integral. This is to represent the constant term which disappears in differentiation, the constant sometimes has the value zero. When we have an *indefinite* integral we always include c but it can usually be given a value in applications.

8.2 Rules of Integration

Basic formula for integration

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c \qquad n \neq -1$$

As with differentiation, there are standard integrals. These are simply the reverse of the standard differentials (see page 36). There are generally more standard results for integrals but this is because they are used more often in applications.

eg $\int \cos(2x + 3)\,dx = 1/2 \sin(2x + 3) + c$

$\int \cos(nx + a)\,dx = 1/n \sin(nx + a) + c$

$\int \sin(2x + 3)\,dx = -1/2 \cos(2x + 3) + c$

$\int \sin(nx + a)\,dx = -1/n \cos(nx + a) + c$

Integrate with respect to x the following .

$x^4 - 5x^5$	$3e^x$	$3\cos(2x + 5)$
....	....	....
....	....	....

[Solutions: $x^5/5 - 5x^6/6$, $3e^x$, $3/2 \sin(2x + 5)$]

Sometimes you will have to change the form of the function to be integrated.

eg $\int (x + 1)^3\,dx = \int x^3 + 3x^2 + 3x + 1\,dx$

$= x^4/4 + 3x^3/3 + 3x^2/2 + x + c$

$\int \cos^2 x\,dx = \int 1/2(1 + \cos 2x)\,dx$

$= 1/2(x + 1/2 \sin 2x + c)$

$= x/2 + 1/4 \sin 2x + c$

How do we evaluate c? If c is left arbitrary then the graph of the function will actually be a set of graphs with varying values of c. However if any conditions are stated then it is quite simple to find c.

eg Integrate $f(x) = 3x^2$ given that $F(1) = 2$

$\int 3x^2\,dx = x^3 + c$

$F(x) = x^3 + c$ so if $F(1) = 1 + c = 2$
then $c = 1$

So $F(x) = x^3 + 1$

Note that $F(x)$ means $\int f(x)$
like $f'(x)$ means $d/dx\, f(x)$

8.3 The Definite Integral

Usually we are looking to *evaluate* an integral, not to just find another equation.
We do this by putting limits on the integral.

eg The velocity of a particle P is given by $V(t) = 3t^2 + 1$ where t is time in seconds.

We can find the distance travelled by P by integrating V(t) as below.

$$\int 3t^2 + 1\, dt = t^3 + t + c$$

This does not tell us anything specific so we put limits on the integral, say we want to know the distance travelled between time t = 1 and t = 4.

$$\begin{aligned} \int_1^4 3t^2 + 1\, dt &= [t^3 + t + c]_1^4 \\ &= [(4)^3 + 4 + c] - [(1)^3 + 1 + c] \\ &= 64 + 4 + c - 1 - 1 - c \\ &= 66 \end{aligned}$$

This is known as the **definite integral**. Notice that the constant disappears so it is not a problem any more.

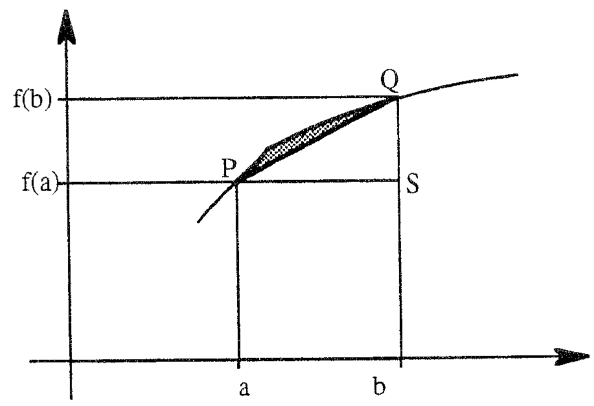
In general, the definite integral between the limits a and b is

$$\begin{aligned} \int_a^b f(x)\, dx &= [F(x)]_a^b \\ &= F(b) - F(a) \end{aligned}$$

8.4 Area Under a Curve

In the same way that differentiation can be looked at graphically, so can integration. Consider the curve f(x) below with points P and Q.

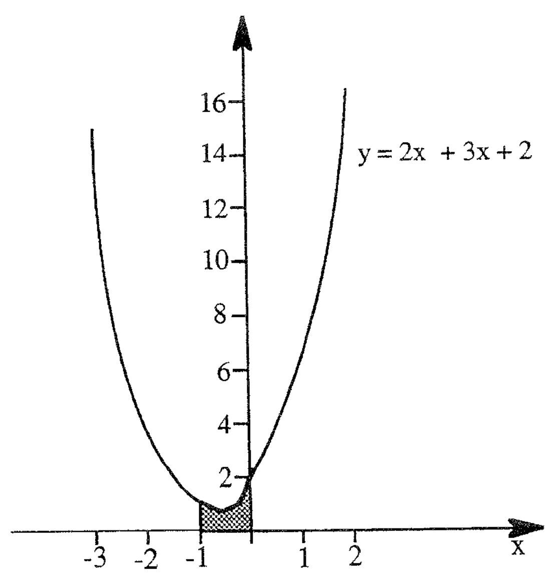
The area under a curve



The area beneath the curve is the sum of a rectangle, a triangle and the shaded area. This, however is not easy to calculate do to the shape of the shaded area. If the whole region is 'sliced' into an infinite number of strips, then the area under the curve is the sum of the areas of all the strips. These strips are all similar in shape to the complete region. However, since they are so narrow, the curved edges can be treated as though they were straight and the error between the actual area and the computed area is negligible. Then each slice has area equal to the sum of a rectangle and a triangle. So the sum of all such strips is the area of the whole region beneath the curve between a and b.

eg Find the area under the curve $y = 2x^2 + 3x + 2$ between $x = -1$ and $x = 0$.

Integration between limits finds the area under a curve between those limits.



The area we require is the shaded area and this is calculated as below.

$$\int_{-1}^{0} 2x^2 + 3x + 2\,dx = [2x^3/3 + 3x^2/2 + 2x]_{-1}^{0}$$
$$= -[-2/3 + 3/2 - 2]$$
$$= 7/6$$

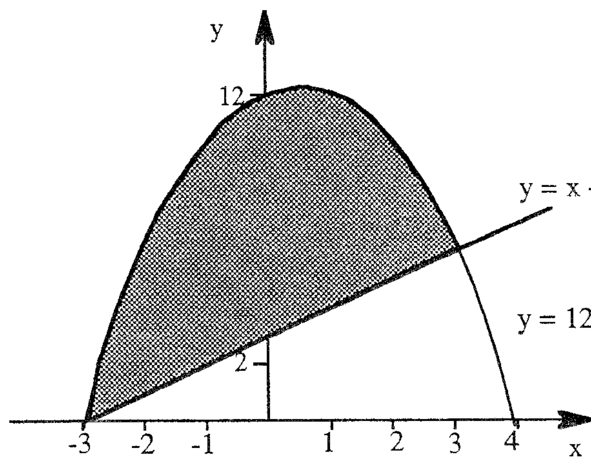
eg Find the area enclosed between the two curves $y = 12 + x - x^2$ and $y = x + 3$.

First, solve the equations simultaneously so that we know the limits of the region.

$$x + 3 = 12 + x - x^2$$
$$x^2 - 9 = 0$$
$$x = \pm 3$$

The co-ordinates of the limits are at (3, 6) and (-3, 0). You can see this on the graph below where the shaded area is the area required.

Integration to find the area between two functions.



Notice that the area between the curves is the difference between the area under $y = 12 + x - x^2$ and the area under $y = x + 3$ between -3 and 3.

ie $$\int_{-3}^{3} 12 + x - x^2\,dx = [12x + x^2/2 - x^3/3]_{-3}^{3}$$
$$= [31.5] - [-22.5]$$
$$= 54$$

$$\int_{3}^{3} x + 3\,dx = [x^2/2 + 3x]_{-3}^{3}$$
$$= [13.5] - [-4.5]$$
$$= 18$$

So the shaded area is 54 - 18 = 36.

8.5 Techniques of Integration.

It is useful to know the following results.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \qquad a \le b \le c$$

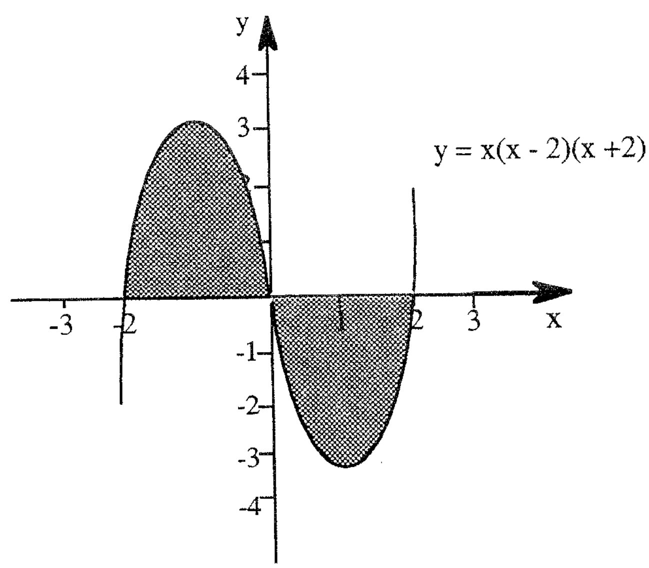
$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \{f(x) - g(x)\}\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

eg Find the area bounded by the curve $y = x(x^2 - 4)$ and the x axis.

$y = x(x + 2)(x - 2)$ and a quick sketch can easily be drawn since the curve crosses the x axis at -2, 0 and 2. The shaded area is the required area.

Integration of a function above and below the x axis



To evaluate this integral we need to split the range into two parts and add their areas. This is because we have an equal amount of area above and below the x axis. To integrate over the entire range -2 to 2 would result in a solution of zero.

$$\int_{-2}^{0} x^3 - 4x\,dx = [x^4/4 - 2x^2]_{-2}^{0} = 4$$

$$\int_{0}^{2} x^3 - 4x\,dx = [x^4/4 - 2x^2]_{0}^{2} = -4$$

The shaded part below the axis has a negative-valued area. This is always the case when the required area is below the axis. The actual area of the shaded region is 8.

Points to remember when evaluating a definite integral, I, are:

- If the area is above the x axis I is positive.
- If the area is below the x axis I is negative.
- f(x) must be defined for *all* values $a \le x \le b$.

In the previous unit we saw that d/dx ln x = 1/x. So clearly,

$$\int 1/x \, dx = \ln | x | + c$$

The following results are extensions of this.

$$\int \frac{1}{ax + b} \, dx = \frac{1}{a} \ln | ax + b | + c$$

$$\int \frac{f(x)}{f(x)} \, dx = \ln f(x) + c$$

$$\int f^{'}(x) \, e^{f(x)} \, dx = e^{f(x)} + c$$

eg $$\int_{-3}^{-1} \frac{1}{2x - 1} \, dx = 1/2[\ln | 2x - 1 |]_{-3}^{-1} + c$$

$$= 1/2 \ln |-3| - 1/2 \ln |7| + c$$
$$= 1/2 \ln 3 - 1/2 \ln 7 + c$$
$$= 1/2 \ln 3/7 + c$$

eg $$\int_1^2 \frac{1}{2x^2 + 3x + 1} \, dx = \int_1^2 \frac{1}{2x+1} - \frac{1}{x+1} \, dx$$ using partial fractions

$$= \left[\frac{2}{2} \ln | 2x + 1| - \ln |x + 1| \right]_1^2 + c$$
$$= \left[\ln \left| \frac{2x + 1}{x + 1} \right| \right]_1^2 + c$$
$$= \ln 5/3 - \ln 3/2 + c$$
$$= \ln 10/9$$

eg $\int x \, e^x \, dx$

This integral has $f(x) = x^2$ and $f^{'}(x) = 2x$, so

$$\int x \, e^x \, dx = 1/2 \int 2x \, e^x \, dx$$
$$= 1/2 \, e^x + c$$

eg $\int_{-2}^{-3/2} \frac{x^2}{x^3 + 1} dx$ has $f(x) = x^3 + 1$ and $f'(x) = 3x^2$, so

$$\int_{-2}^{-3/2} \frac{x^2}{x^3 + 1} dx = 1/3 \int_{-2}^{-3/2} \frac{3x^2}{x^3 + 1} dx$$
$$= 1/3[\ln |x^3 + 1|]_{-2}^{-3/2}$$
$$= 1/3 \ln |19/8| - 1/3 \ln |7|$$
$$= 1/3 \ln 19/56$$

Substitution

This is essentially the reverse of differentiation using the function of a function method. Notice that by using the substitution u for x we can write du as $\frac{du}{dx} dx$.

eg Find $\int (3x + 1)^3 dx$

Let $u = 3x + 1$ since the integral $\int u^3$ is a standard technique.

$$\frac{du}{dx} = 3$$

Now, $f(x) = (3x + 1)^3 = 1/3 (3x + 1)^3 \frac{du}{dx} = g(u) \frac{du}{dx}$

So, $g(u) = 1/3 (3x + 1)^3 = 1/3 u^3$

Then $\int f(x) dx = 1/3 \int u^3 \frac{du}{dx} dx$
$$= 1/3 \int u^3 du$$
$$= u^4/12 + c$$
$$= 1/12 (3x + 1)^4 + c$$

eg Integrate $y = \frac{1}{(3x - 4)^4}$ with respect to x.

Let $u = 3x - 4$ so $du/dx = 3$ and $y = 1/u^4$

So $\int \frac{1}{(3x - 4)^4} dx = 1/3 \int u^{-4} \frac{du}{dx} dx$

$= 1/3 \int u^{-4} du$

$= 1/3 (-1/3\ u^{-3}) + c$

$= -1/9u^3 + c$

$= -\frac{1}{9(3x - 4)^3} + c$

eg Integrate with resect to x, $y = \frac{1}{3x - 4}$

Put u = 3x - 4 as before, so that dx = 1/3 du and y = 1/3 .1/u . du

Then, $1/3 \int 1/u\ du = 1/3 \ln|u| + c$

$= 1/3 \ln|3x - 4| + c$

eg Integrate y = sin (3x - 4) dx with respect to x.

Let u = 3x - 4 so that dx = 1/3 du and y = sin u

Then, $1/3 \int \sin u\ du = -1/3 \cos u + c$

$= -1/3 \cos (3x - 4) + c$

When the method of substitution is used for definite integrals, we must be sure to change the limits as well as the function.

$\int_a^b f(x)\ dx = \int_{u(a)}^{u(b)} g(u)\ du$ where f(x) = g(u) du/dx

eg Evaluate $\int_0^1 \frac{x^2}{1 + x^3} dx$

Let $u = 1 + x^3$ so that $dx = 1/3x^2\ du$

Then, $\int_0^1 \frac{x^2}{1 + x^3} dx = \int_{u(a)}^{u(b)} \frac{1}{3u} du$

$= 1/3 \int_1^2 1/u\ du$

$= 1/3 \ln|2| - 1/3 \ln|1| + c$

$= 1/3 \ln 2 + c$

Notice that we find u(a) and u(b) by substituting a and b into u.

eg Evaluate $I = \int_0^3 x\sqrt{(x + 1)}\, dx$

Let $u = \sqrt{(x + 1)}$ so that $x = u^2 - 1$

$\frac{du}{dx} = 1/2\, (x + 1)^{-1/2} = 1/2u$ then $dx = 2u\, du$

So, $I = \int_{u(0)}^{u(1)} (u^2 - 1)\, u\, 2u\, du$

$= \int_1^2 2u^2(u^2 - 1)\, du$

$= \int_1^2 2u^4 - 2u^2\, du$

$= [2u^5/5 - 2u^3/3 + c]_1^2$

$= [2/5\,(32) - 2/3\,(8)] - [2/5 - 2/3]$

$= 116/15.$

It is helpful to look for u(x) and g(u) so that $f(x) = g(u)\, u^{I}(x)$.

If there is a square root involved in the function it is useful to replace it with a new variable.

eg Integrate the following $\int \frac{3x^2}{\sqrt{(1 + x^3)}}\, dx$

Let $u = \sqrt{(1 + x^3)}$ so that $u^2 = 1 + x^3$

$2u \frac{du}{dx} = 3x^2$ Implicit differentiation

Then, $\int \frac{3x^2}{\sqrt{(1 + x^3)}}\, dx = \int \frac{2u}{u} \frac{du}{dx}\, dx$

$= \int 2\, du$

$= 2u + c$

$= 2(1 + x^3)^{1/2} + c$

We know that $d/dx \ln | g(x) | = \frac{g^{'}(x)}{g(x)}$

So it helps in the working out to find a substitution u = g(x) if the function has this form.

eg Find $\int \frac{1}{\sin x \cos x} dx$

Now, $\frac{1}{\sin x \cos x} = \frac{\sec^2 x}{\tan x} = \frac{g^1(x)}{g(x)}$

So let $u = \tan x \quad du/dx = \sec^2 x$

Then, $\int \frac{1}{\sin x \cos x} dx = \int \frac{\sec^2 x}{\tan x} dx = \ln | \tan x | + c$

Integrating Rational Functions

A rational function has the form $f(x) = \frac{p(x)}{q(x)}$ where p(x) and q(x) are polynomials.

This type of function *must* be split into partial fractions, otherwise you will not be able to integrate it at all. The case when this is not neccessary is when $p(x) = q^1(x)$ as we saw earlier.

ie $\int \frac{q^1(x)}{q(x)} dx = \ln | q(x)| + c$

There are a number of standard cases where we have the general integrals below.

$$\int \frac{1}{ax + b} dx = 1/a \ln | ax + b | + c$$

$$\int \frac{1}{(ax + b)^2} dx = - \frac{1}{a(ax + b)} + c$$

$$\int \frac{a}{x^2 + a^2} dx = \tan^{-1} x/a + c$$

These are derived using the method of substitution. How? That is not shown but it is quite easy really. The important thing is, that if you can remember them you will save time in your calcualtions.

eg Find $\int_0^1 \frac{x^2}{x^2 + 3x + 2} dx$

Splitting into partial fractions we arrive at the integral below.

$$\int_0^1 \frac{x^2}{x^2+3x+2}\,dx = \int_0^1 1 - \frac{3x+2}{(x+2)(x+1)}\,dx$$
$$= \int_0^1 1\,dx - \int_0^1 \frac{4}{x+2}\,dx + \int_0^1 \frac{1}{x+1}\,dx$$
$$= [x - 4\ln|x+2| + \ln|x+1|]_0^1$$
$$= 1 + \ln 32/81$$

eg Find $\int_0^1 \frac{x+1}{x^2+x+1}\,dx$

We cannot split the denominator into factors in this case, so we complete the square and proceed in that way.

$$\int_0^1 \frac{x+1}{x^2+x+1}\,dx = \int_0^1 \frac{x+1/2+1/2}{(x+1/2)^2+3/4}\,dx$$

Let $u = x + 1/2$ so $dx = du$

Then $u(0) = 1/2$, $u(1) = 3/2$

$$\int_0^1 \frac{x+1/2+1/2}{(x+1/2)^2+3/4}\,dx = \int_{1/2}^{3/2} \frac{u+1/2}{u^2+3/4}\,dx$$
$$= \frac{1}{2}\int_{1/2}^{3/2} \frac{2u}{u^2+3/4}\,du + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\int_{1/2}^{3/2} \frac{\sqrt{3}/2}{u^2+3/4}\,du$$
$$= [1/2 \ln|u^2+3/4| + 1/\sqrt{3}\tan^{-1}\frac{2u}{\sqrt{3}}]_{1/2}^{3/2}$$
$$= \ln\sqrt{3} + \frac{\pi}{6\sqrt{3}}$$

In general, this form of equation is written

$$\int \frac{cu+d}{u^2+b^2}\,du = c/2 \ln|u^2+b^2| + d/b \tan^{-1}(u/b)$$

Integration by Parts

When the integrand under consideration is the product of two functions and it is not in any of the standard forms already mentioned, we use the following formula which is derived from the formula for the differential of the product of two functions.

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$$

where f(x) = u and g(x) = v.

In order to do this we need to choose u and v so that v du/dx is easier to integrate than u dv/dx. A simple way to choose u is by the **LATE** rule of priority.

High Priority ↑	**L**ogarithm
	Algebra
	Trigonometry
Low Priority ↓	**E**xponential

eg Find $\int x \ln x \, dx$

$$u = \ln x \qquad \frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = x^2/2$$

$$\int x \ln x \, dx = x^2/2 \ln x - 1/2 \int x^2/x \, dx$$
$$= x^2/2 \ln x - x^2/4 + c$$

eg Find $\int xe^{3x} \, dx$

$$u = x \qquad \frac{dv}{dx} = e^{3x}$$

$$\frac{du}{dx} = 1 \qquad v = e^{3x}/3$$

$$\int x\, e^{3x} \, dx = xe^{3x}/3 - 1/3 \int e^{3x} \, dx$$
$$= xe^{3x}/3 - e^{3x}/9 + c$$

eg Evaluate $\int_0^{\pi/2} \cos^4 x \, dx$

$u = \cos^3 x \qquad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = -3 \sin x \cos^2 x \qquad v = \sin x$

$$
\begin{aligned}
I = \int_0^{\pi/2} \cos^4 x \, dx &= [\cos^3 x \sin x]_0^{\pi/2} + 3 \int_0^{\pi/2} \sin^2 x \cos^2 x \, dx \\
&= 0 - 0 + 3 \int_0^{\pi/2} (1 - \cos^2 x) \cos^2 x \, dx \\
&= 3 \int_0^{\pi/2} \cos^2 x \, dx - 3I \\
4I &= 3 \int_0^{\pi/2} \cos^2 x \, dx \\
&= 3/2 \int_0^{\pi/2} (1 + \cos 2x) \, dx \\
&= 3/2 \, [x + 1/2 \sin 2x]_0^{\pi/2} \\
&= 3\pi/4
\end{aligned}
$$

So $I = 3\pi/16$

Notice here that we use the letter I to denote the integrand we are looking for.
This enables us to see clearly how the function integrates and avoids the feeling of going round in circles. When integrating trigonometric functions this is a useful method to know.

Summary

We have seen that the technique of integration is the reverse of differentiation and as such it has a number of standard integrals. There are an infinite number of integrals which are called 'standard' but the ones which you are most likely to use are the most important ones. These are listed below.

$f(x)$	$\int f(x)\,dx$
x^n	$\frac{x^{n+1}}{n+1}$
$1/x$	$\ln \lvert x \rvert$
e^x	e^x
ae^x	e^x/a
a^x	$a^x/\ln a$
xe^x	$xe^x - e^x$
$\ln x$	$x \ln x - x$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\ln \lvert \sec x \rvert$
$\sin^2 x$	$x/2 - 1/4 \sin 2x$
$\cos^2 x$	$x/2 + 1/4 \sin 2x$
$x \sin x$	$- x \cos x + \sin x$
$x \cos x$	$x \sin x + \cos x$

There are a great many more, and as you come across them you will get to know those which you are most likely to use again. It is important to remember that the results for many standard cases can be found by *doing* the integration so you have to decide for yourself whether you want to remember vast numbers of identities or become better acquainted with integration techniques.

Integration is fundamentally a method of finding the area under a curve but because of the nature of the mathematics, we need to know the various techniques of integration. Sometimes you may find that there are two different ways of integrating the same function, as long as they both give the same solution it does not matter which way you do it.

Activities

1. Integrate the following.

(a) $\int 6x^2\,dx$ (b) $\int 4x - 2\,dx$ (c) $\int \cos 2x\,dx$
(d) $\int 3x/2\,dx$ (e) $\int -2/x^2\,dx$ (f) $\int 5/x^2 + 1\,dx$
(g) $\int (x^2 + 1)(x^2 - 1)\,dx$ (h) $\int (\cos x + \sin x)^2$ (i) $\int ax^2 + bx + c\,dx$
(j) $\int 2x^{-1/2} + 3x^{1/2}\,dx$ (k) $\int \sin 2x\,dx$ (l) $\int (s + 4)^3\,ds$

2. A curve has the equation given by f(x). Find f(x) given that

(i)	$dy/dx = 4x$	curve passes through (0, 4)
(ii)	$dy/dx = 3x^2 - 6x + 3$	(1, 0)
(iii)	$dy/dx = 3(x + 2)^2$	(0, 8)
(iv)	$dy/dx = 2\cos 2x - 2\sin 2x$	(0, 1)

3. Evaluate

(a) $\int_1^3 2t + 4\,dt$ (b) $\int_0^{\pi/2} \sin 2x\,dx$ (c) $\int_1^4 3x^2 - 2x\,dx$
(d) $\int_{-2}^2 2 + x^2\,dx$ (e) $\int_1^4 \sqrt{x}\,dx$ (f) $\int_{-2}^{-1} 2x^2 - x^3\,dx$
(g) $\int_0^{\pi/4} \sin x\,dx$ (h) $\int_{\pi/6}^{\pi/3} \cos x\,dx$ (i) $\int_2^3 1/\sqrt{x}\,dx$

4. Draw a graph to illustrate the area given by the interals below and evaluate them.

(a) $\int_1^3 2x + 4\,dx$ (b) $\int_1^2 3 - x\,dx$ (c) $\int_{-1}^1 4 - 2x^2\,dx$ (d) $\int_2^4 \sqrt{x}\,dx$

5. Find the area between the two equations given.

(a) $y = 3x + 5$, $y = x^2 + 1$ (b) $y = 3 - x^2$, $y = 2x^2$ (c) $y = x^2$, $y = 3x$

6. Evaluate the folowing definite integrals and dislpay them on a graph.

(a) $\int_{-\pi/2}^{\pi/2} \sin x\,dx$ (b) $\int_{\pi/2}^{3\pi/2} \cos x\,dx$ (c) $\int_{\pi/3}^{2\pi/3} \sin 3x\,dx$

7. Find

(a) $\int \frac{2}{x^2 + 5x + 6}\,dx$ (b) $\int \frac{1}{x^2 + 3x}\,dx$

(c) $\int \frac{4}{(x + 1)^2}\,dx$ (d) $\int \frac{1}{2 - x}\,dx$ (e) $\int \frac{3}{4x + 1}\,dx$

(f) $\int \frac{-2}{x + 7}\,dx$ (g) $\int_{-2}^{-1} \frac{3}{4x + 1}\,dx$ (h) $\int_{-10}^{-8} \frac{-2}{x + 7}\,dx$

8. Integrate the following functions with respect to x.

(a) $y = x^2e^{2x}$ (b) $y = e^{\tan x} \sec^2 x$ (c) $y = x^{-2}\, e^{1/x}$

(d) $y = x^{-1/2}\, e^{x/2}$ (e) $y = e^{1/x} \,/\, x^3$ (f) $y = e^{\cot x} / \sin^2 x$

9. Evaluate $\int_0^1 \frac{e^x}{1 + e^x}\,dx$

10. Use the substitution given to find the integrals below.

(a) $\int (x + 2)^2\,dx$ $u = x + 2$ (b) $\int \sqrt{(3x - 4)}\,dx$ $u = 3x - 4$

(c) $\int 2 \sin (3x + 4)\,dx$ $u = 3x + 4$ (d) $\int x(1 - x^2)_3\,dx$ $u = 1 - x^2$

(e) $\int x\sqrt{(x^2 + 2)}\,dx$ $u^2 = x^2 + 2$ (f) $\int \sin x \cos^4 x\,dx$ $u = \cos x$

(g) $\int (x^2 + 1)^{-3/2}\,dx$ $x = \tan \vartheta$ (h) $\int 1/(4 + x^2)\,dx$ $x = 2 \tan \vartheta$

11. Integrate the following functions.

(a) $y = 4(3 - 2x)^3$ (b) $y = (x - 1)^{-1/2}$ (c) $y = \sin^5 x \cos x$

12. Evaluate

(i) $\int_6^{\pi/2} \cos(x/2 + 3\pi/4)\,dx$ (ii) $\int_1^2 (4x - 3)^3\,dx$ (iii) $\int_0^4 x\sqrt{(2x + 1)}\,dx$

(iv) $\int_0^4 3x\sqrt{(x^2 + 9)}\,dx$ (v) $\int_1^e (1 + x^2)/x\,dx$ (vi) $\int_0^1 x\, e^{-x}\,dx$

(vii) $\int_0^2 \frac{x^2}{\sqrt{(x^3 + 1)}}\,dx$ (viii) $\int_{\sqrt{2}}^{\sqrt{5}} \frac{x^3}{\sqrt{(x^2 - 1)}}\,dx$ (ix) $\int_3^4 \frac{x - 1}{\sqrt{(x^2 - 2x)}}\,dx$

13. Integrate the following.

(a) $\int x^2 \sin x\,dx$ (b) $\int e^{-x} \cos(2x + 3)\,dx$ (c) $\int x^2\, e^x\,dx$

(d) $\int x \cos nx\,dx$ (e) $\int x \sin x \cos x\,dx$ (f) $\int x^{-1} \ln x\,dx$

14. Evaluate the following.

(a) $\int_0^2 x e^{-x}\,dx$ (b) $\int_0^{\pi} x^3 \cos x\,dx$ (c) $\int_0^{\pi} x \sin 2x\,dx$

(d) $\int_0^1 xe^{-2x}\,dx$

15. Evaluate.

(a) $\int_0^3 \frac{4(x+5)}{(x+1)^2(x^2+3)}\,dx$ (b) $\int_3^{11} \frac{x}{(x+1)(x-2)}\,dx$

(c) $\int_3^4 \frac{x^3+4x+4}{(x-2)^2(x^2+1)}\,dx$ (d) $\int_{-1}^1 \frac{e^x}{e^x+1}\,dx$

(e) $\int_0^2 \frac{x}{(x+2)^2}\,dx$ (f) $\int \frac{x}{(1+x)^2}\,dx$

[Solutions: **1** (a) $2x^3 + c$ (b) $2x(x + 1) + c$ (c) $1/2 \cos 2x + c$ (d) $3x^2/4 + c$ (e) $2/x + c$ (f) $-5/x + x + c$

(g) $x^5/5 + x + c$ (h) $x - 1/2 \cos 2x + c$ (i) $ax^3/3 + bx^2/2 + cx + d$

(j) $4x^{1/2} + 2x^{3/2} + c$ (k) $-1/2 \cos 2x + c$ (l) $s^4/4 + 4s^3 + 24s^2 + 64s + c$;

2 (i) $f(x) = 2x^2 + 4$ (ii) $f(x) = x^3 - 3x^2 + 3x - 1$ (iii) $f(x) = x^3 + 6x^2 + 12x + 8$

(iv) $f(x) = \sin 2x + \cos 2x$;

3 (a) 16 (b) 1 (c) 48 (d) 40/3 (e) 14/3 (f) 101/12 (g) $1 - 1/\sqrt{2}$ (h) $(\sqrt{3}-1)/2$ (i) 1/6;

4 (a) 16 (b) 3/2 (c) 20/3 (d) $(16 - 4\sqrt{2})/3$;

5 (a) 125/6 (b) 4 (c) 4.5;

6 (a) 0 (b) -2 (c) -2/3;

7 (a) $2 \ln [|x+2|/|x+3|] + c$ (b) $1/3 \ln |x / x+3| + c$ (c) $-4(x+1)^{-1} + c$

(d) $-\ln |2 - x| + c, x \neq 2$ (e) $3/4 \ln |4x + 1| + c, x \neq -1\backslash 4$ (f) $-2 \ln |x + 7| + c, x \neq -7$

(g) $3\backslash 4 \ln 3\backslash 7$ (h) $2 \ln 3$;

8 (a) $1\backslash 6\, e^{2x} + c$ (b) $e^{\tan x} + c$ (c) $-e^{1/x} + c$ (d) $2e^{x/2} + c$ (e) $-1/2\, e^{1/x} + c$ (f) $-e^{\cot x} + c$;

9 $\ln 1/2(1 + e)$;

10 (a) $1/3\,(x + 2)^3 + c$ (b) $2/9\,(3x - 4)^{3/2} + c$ (c) $-2/3 \cos (3x + 4) + c$

(d) $-1/8\,(1 - x^2)^4 + c$ (e) $1/3\,(x^2 + 2)^{3/2} + c$ (f) $c - 1/5 \cos^5 x$ (g) $x / (1 + x^2) + c$

(h) $1/2 \tan^{-1} x/2 + c$;

11 (i) $c - 1/2(3 - 2x)^4$ (ii) $2(x - 1)^{1/2} + c$ (iii) $1/6 \sin^6 x + c$ (iv) $1/2 \tan (2x - 1) + c$;

12 (i) $-\sqrt{2}$ (ii) 39 (iii) 298/15 (iv) 98 (v) $1/2\,(e^2 + 1)$ (vi) $1/2\,(1 - e^{-1})$ (vii) 4/3

(viii) $\sqrt{8} - \sqrt{3}$ (ix) 10/3;

13 (a) $(2 - x^2) \cos x + 2 \sin x + c$ (b) $e^{-x}/5\,[2 \sin(2x + 3) - \cos (2x + 3)] + c$

(c) $e^x(x^2 - 2x + 2) + c$ (d) $1/n^2\,(nx \sin nx + \cos nx) + c$

(e) $1/2\, x \sin^2 x + 1/8 \sin 2x - x/4 + c$ (f) $1/2\,(\ln|x|)^2 + c$;

14 (a) $1 - 3e^{-2}$ (b) $12 - 3\pi^2$ (c) $-\pi/2$ (d) $1/4 - 3/4\, e^{-2}$;

15 (a) $3(1 + \ln 2) - 13\pi/9$ (b) $5/3 \ln 3$ (c) $2 + 1/2 \ln 17/10$ (d) 1 (e) $\ln 2 - 1/2$ (f) $\ln |1 + x| + 1/(1 + x) + c$.]